Essays in
Econometrics of Financial Asset Pricing Models

Dissertation
for the Faculty of Economics, Business Administration and Information Technology of the University of Zurich

to achieve the title of
Doctor of Philosophy
in Banking & Finance

presented by
Mustafa Arif Karaman
from Turkey

approved in April 2012 at the request of
Prof. Dr. Loriano Mancini
Prof. Dr. Markus Leippold

The Faculty of Economics, Business Administration and Information Technology of the University of Zurich hereby authorises the printing of this Doctoral Thesis, without thereby giving any opinion on the views contained therein.

Zurich, April 4, 2012.

Chairman of the Doctoral Committee: Prof. Dr. Dieter Pfaff

Aile'me.

Acknowledgments

I cannot find enough words to express my gratitude to my advisor Professor Loriano Mancini. I would like to thank him for his guidance, insightful discussions, warm encouragements, and support to overcome the crises throughout my Ph.D. studies. From him I learned not only excellence in research but also in personality.

I would like to thank Professor Markus Leippold for accepting to be a member of my thesis committee.

I am indebted to Professor Yacine Aït-Sahalia, Professor Marc Chesney, and Doctor Ganna Reshetar for allowing me collaborate with them and learn from them.

I would like to thank Professor Mete Soner for many interesting discussions on various financial topics.

I would like to thank Professor Ramazan Gençay for his constant support since the early stages of my Ph.D. studies and many long conversations on finance and econometrics.

I would also like to thank my friends, Mehmet Ali Yatbaz, Gürkan Sarıoğlu, Hakan Karadut, Murat Temir, Korcan Dericioğlu, Korman Barlas, Can Kut, Alper Odabaşıoğlu, Ethem Güney, Erdinç Akyıldırım, Hasan Yaman, Emre Atsan, Ozan Akdoğan, and Evran Alper for their help enjoy my free time during the graduate studies.

My family, above all, has always been an endless source of support throughout my whole studies. To them I dedicate this thesis.

Contents

Part I

Introduction and Summary of Results

1. Introduction

This dissertation deals with econometrics of financial asset pricing models. In the following there parts some specific asset pricing models in finance are first analyzed theoretically and then introduced to financial data aiming to answer how investors price various types of risks and how they incorporate those risks into their financial decisions. In particular, in the first part the focus is on the analysis of terrorism risk via discrete-time financial asset pricing models whereas in the second and third parts the variance swap market is studied in the continuous-time modeling framework. Three research papers that constitute this dissertation are:

(i) The Impact of Terrorism on Financial Markets: An Empirical Study (with Marc Chesney, and Ganna Reshetar),

(ii) Variance Swaps, Risk Premiums, and Expectation Hypothesis (with Yacine Aït-Sahalia, and Loriano Mancini),

(iii) Euler Approximation and Likelihood Expansion for Continuous-Time Derivative Pricing Models: A Comparative Analysis.

2. Summary of Results

(i) The Impact of Terrorism on Financial Markets: An Empirical Study

The first research article of this dissertation analyzes how investors price terrorism risk in financial markets and how they incorporate this risk into their decision making process. The 9/11 terrorist attacks and more recent attacks in Madrid in 2004 and in London in 2005 revealed that terrorism risk is a new type of catastrophic risk that investors and financial institutions may be facing. In order to study this risk, we look at the effect of 77 terrorist attacks that occurred in 25 countries over an 11-year (from 1994 to 2005) time period. We look at global, European, American, and Swiss stock markets as well as insurance, banking, travel, pharma/biotech, aero/defense and oil/gas industrial stock indices. In contrast to impact studies which often employ only event-study methodology, in this work we investigate the impact of terrorism using other methods as well, namely a non-parametric methodology and a filtered GARCH with the Extreme Value Theory (EVT) approach. Both methods are standard econometric tools. However, the application of them in this work is original. The analysis shows that a non-parametric approach is the most appropriate method among the three considered for analyzing the impact of terrorism on financial markets. The results of this work show that approximately two-thirds of the terrorist attacks considered lead to a significant negative impact on at least one stock market under consideration. The insurance sector and the airline industry exhibit the highest susceptibility to terrorism, while the banking industry is the least sensitive. This is in contrast to financial crashes which have a strong negative impact on the banking sector. The analysis of the possible diversification strategies considering terrorism as a risk factor for Swiss and European investors shows that an investor would be better of if the terrorism risk is considered in the portfolio allocation decisions.

(ii) Variance Swaps, Risk Premiums, and Expectation Hypothesis

The second research article of this dissertation studies how the term structure of equity, jump and volatility risk premiums behave in variance swap market. In addition, forecasting power of those rates for the future volatility levels, and possible investment strategies considering volatility as an asset class are also analyzed. A variance swap is a forward contract on future realized variance and it is the most direct way to achieve an exposure to or hedge against variance risk. Similar to any other forward contract, the long position receives the floating realized variance over a certain time horizon and pays a fixed rate, called variance swap rate. In the first part of this paper, a model-free analysis of the variance swaps is performed in order to understand the hidden dynamics in the data which is daily variance swap rates (not synthetic rates) on the S&P 500 index with fixed time to maturity of 2-, 3-, 6-, 12- and 24-month from January 4, 1996 to September 2, 2010. This analysis reveals that there are randomly-occurring jumps in the asset returns and the volatility of the returns should include two factors to capture observed term structures. Therefore a parsimonious way to model variance swaps is via a two-factor stochastic volatility and stochastic jump-intensity model which is in the class of affine models in finance. In the second part, this model is fitted to variance swap data via the closed-form likelihood expansion scheme, a recently developed estimation technology in the literature. Intuitively, this methodology approximates the unknown transition density of the stochastic processes by deforming or stretching the Normal density so that it tries to capture the required skewness, kurtosis and the higher moments of the transition density. The two-factor stochastic volatility and stochastic jump-intensity model implies variance swap rates that are affine functions of unobserved state variables like volatility or the long-run mean of the volatility. Therefore, recovering those latent states is straightforward especially compared to the case with extracting those states from option prices. The estimation results show that the term structure of variance risk premium is negative and generally downward sloping, while the term structure of variance risk premium due to negative jumps is negative, downward sloping in quite times and upward sloping during market crashes. Moreover, the forecasting power of variance swap rates for future volatility levels is not strong for long horizons due to the variance risk premium. However bias and inefficiency of this prediction is modest for short/medium time to maturities. Moreover, a comparison of the trading strategy, which is based on shorting a variance swap if the expected profit from this investment is more than a threshold level, appears to be more profitable than investing in the S&P 500 index, over the same time horizons.

(iii) Euler Approximation and Likelihood Expansion for Continuous-Time Derivative Pricing Models: A Comparative Analysis

The third research article of this dissertation compares estimation performance of Euler discretization and closed-form likelihood expansion methodologies for four derivative pricing models. The models studied are nested in the sense that it is possible to reduce the biggest model to the other three with some constraints on the modeling parameters. Therefore, this paper not only compares two different estimation schemes within different models but also analyzes how each model performs with respect to each other to explain the data characteristics. That is, the implied instantaneous market, jump and volatility risk premiums and pricing errors are analyzed both within and between models. In the first part, an extensive Monte Carlo simulation study shows that likelihood expansion outperforms the Euler approximation. In the second part all models are estimated with real-data via both estimation schemes. The data used in this study is the variance swap data. The estimations show that the power of the likelihood expansion over the Euler approximation is more pronounced as the joint dynamics of the state variables are further away from the multivariate normal density. This corresponds to the case where asset returns have jumps. Since it is widely accepted that the asset prices include jumps in addition to continuous Brownian dynamics, the method of estimation should be closed-form likelihood expansion for such models. Moreover, the comparison of the estimates and the recovered quantities reveal that the stochastic-intensity two-factor model fits the variance swap data best.

Part II

The Impact of Terrorism on Financial Markets: An Empirical Study

Marc Chesney, Ganna Reshetar, and Mustafa Karaman

This paper was accepted for publication in the Journal of Banking and Finance, Volume 35, Issue 2, February 2011, Pages 253 - 267.

Abstract

The main focus of this paper is to study empirically the impact of terrorism on the behavior of stock, bond and commodity markets. We consider terrorist events that took place in 25 countries over an 11-year time period and implement our analysis using different methods: an event-study approach, a non-parametric methodology, and a filtered GARCH-EVT approach. In addition, we compare the effect of terrorist attacks on financial markets with the impact of other extreme events such as financial crashes and natural catastrophes. The results of our analysis show that a non-parametric approach is the most appropriate method among the three for analyzing the impact of terrorism on financial markets. We demonstrate the robustness of this method when interest rates, equity market integration, spillover and contemporaneous effects are controlled. We show how the results of this approach can be used for investors' portfolio diversification strategies against terrorism risk.

1. Introduction

A lot of research on terrorism has been done in the fields of sociology, political science and history. With respect to economics and finance, terrorism has not received much attention from researchers until recently. The effect of the impact of the 9/11 terrorist attacks on stock markets as well as that of more recent attacks in Madrid in 2004 and in London in 2005 has revealed that terrorism risk is a new type of catastrophic risk that investors and financial institutions may be facing. In this paper we intend to provide a deeper understanding of the impact of this risk on the behavior of various financial markets. When studying the impact, we look at global, regional, national and industrial market levels. In addition, we compare the impact of terrorist events on financial markets with the impact of other extreme events such as financial crashes and natural catastrophes.

Among existing research, this empirical paper is one of the very few (see for example Arin et al. (2008), Chen and Siembs (2004), Eldor and Melnick (2004), Karolyi and Martell (2006)) that study the link between terrorism and the behavior of stock markets. It is also the first one that analyzes the impact on bond and commodity markets.

In contrast to impact studies which often employ only event-study methodology, in this work we investigate the impact of terrorism using other methods as well. We use a non-parametric methodology and a filtered GARCH with the Extreme Value Theory (EVT) approach. Both methods are standard econometric tools. However, our application of them in this work is original.

We show that a non-parametric approach is the most appropriate method among the three considered for analyzing the impact of terrorism on financial markets. In contrast to an event-study, it does not impose strong parametric restrictions. It is also less computationally intensive than a filtered GARCH-EVT method. Finally, a non-parametric approach allows us to analyze the impact of events in the post-event period which is not possible with the GARCH-EVT approach. We demonstrate the robustness of a non-parametric model when interest rates, equity market integration, spillover and contemporaneous effects are controlled.

The findings of our empirical investigation are useful for investors, insurance and re-insurance businesses, banks and government agencies. This study is the first one to give insights into possible portfolio diversification strategies with respect to the risk of terrorism. We demonstrate how Swiss and European investors can apply the results of our non-parametric model to the construction of their investment portfolios.

In order to study the impact of terrorism on financial markets empirically, we look at the effect of 77 terrorist attacks that occurred in 25 countries over an 11-year time period. We look at global, European, American, and Swiss stock markets as well as insurance, banking, travel, pharma/biotech, aero/defense and oil/gas industrial stock indices. In relation to other markets, we look at the US, European and World bond indices as well as at the global commodity and gold markets. When comparing the impact of terrorist events on these financial markets with the impact of other extreme events, we analyze the impact of 4 financial crashes and 19 natural catastrophes which occurred during the 11-year period under consideration.

The results of our work are as follows. Approximately two-thirds of the terrorist attacks considered lead to a significant negative impact on at least one stock market under consideration. The Swiss stock market is affected by the highest number of attacks while the American stock market by the lowest number. The insurance sector and the airline industry exhibit the highest susceptibility to terrorism, while the banking industry is the least sensitive.[1] This is in contrast to financial crashes which have a strong negative impact on the banking sector. The analysis of the impact on the aero/defense, pharma/biotech and oil/gas sectors shows both positive and negative reactions. These sectors behave similarly to natural disasters and financial crashes.

As with terrorist events, natural catastrophes cause both positive and negative return movements in the commodity/gold and bond markets. The gold index is affected by a lower number of events compared to the commodity index, implying less sensitivity of the former to natural disasters. Finally, among bond markets considered, the US government bond market shows the lowest impact from terrorist attacks, natural catastrophes and financial crashes.

As to the strength of the impact, terrorist attacks and financial crashes cause event-day return movements that are mostly extreme, with the strength of the impact declining in the post-event period. This implies that although markets perceive these events as unusual, they do not see their effects as long-lasting. Regarding natural catastrophes, the negative impact is more often observed in the post-event period. This can be attributed to the fact that markets need more time to evaluate the long-term impact of such events. Furthermore, as natural disasters can last for several days, the impact is more likely to be evaluated during the period following the event.

The results of this paper suggest several diversification strategies for minimizing the terrorism risk. Investors concerned about this risk should consider holding two groups of assets: those

[1] Note that the banking sector was affected negatively by the 9/11 attacks. However, this event was exceptional in terms of its magnitude and place of occurrence (Manhattan, the financial center).

which are likely to react positively to terrorist attacks, or those which have little or no negative sensitivity to this risk. In the first case, a US Government bond index is the safest choice followed by such industry stocks as aero/defense and pharma/biotech. However, given that these stock markets may also exhibit a negative response, investing in these industries as a diversification strategy against terrorist attacks may not always work. In the second case, a banking stock index may be good for investment. It is important to note, however, that although a banking stock index is least sensitive to terrorist attacks, it exhibits significant negative return movements associated with financial crashes.

Regarding the other financial markets, investing in a composite commodity index is preferable to investing in gold only. This is because the gold market often reacts more negatively than positively to terrorist events. In addition, when compared to the commodity market in general, the negative impact on the gold market is more long-lasting. At the same time, the commodity market also shows a short-term negative reaction to some terrorist events. This implies that investing in gold and commodity markets may not always provide a good hedge.

Another possible way to reduce negative exposure to terrorist events would be to avoid investing in insurance, travel and airline stocks or to short these indices. Note that insurance and airline industries show high negative sensitivity not only to terrorist attacks but also to financial crashes and natural disasters. This implies that by taking long positions in these stocks, investors may end up increasing their risk of loss if further terrorist attacks occur.

Finally, our analysis of the possible investment diversification strategies for Swiss and European investors shows that an investor who uses the results of our paper and constructs an investment portfolio in such a way that she imposes a negative correlation on the industries that react inversely to the terrorist attacks and a positive average correlation on the rest of the covariance matrix, would outperform other investors.

2. Related research

Analysis of existing literature on the impact of terrorism on financial markets shows that most of the research has a descriptive character and focuses on the impact of very few terrorist events (often only those which occurred on September 11, 2001). A recent article by Karolyi (2006) discusses what is known and unknown about the effects of terrorist events on financial markets. It also provides a summary of the research done in this area. According to the author, there is still little known about the economic and financial consequences of terrorism.

A very recent paper by Arin et al. (2008) shows interesting results regarding the effect of terrorist events on the markets' behavior based on evidence from six different financial markets (Indonesia, Israel, Spain, Thailand, Turkey and UK). In their work, the authors investigate the effects of terrorism not only on the stock markets, but also on stock market volatility. They find that the magnitude of terrorist effects is larger in emerging markets.

Johnston and Nedelescu (2005) examine cases in which financial markets are directly or indirectly affected by terrorist acts. They review the reaction of the markets to the 9/11 attacks in the US and the attacks in Madrid in March, 2004. The main conclusion of their study is that financial markets are not only confronted with major disruptions caused by the massive damage to property and communication systems, but also with high levels of uncertainty and market volatility, especially in the case of the 9/11 attacks in New York. However, there are some differences in the stock market reaction to these two terrorist events. While the attacks in Madrid were perceived as mostly having a regional effect, those in New York were seen as having repercussions on the global financial system.[2] The authors view the timing of the attacks as a possible explanation for the different impacts. Whereas the attacks in New York occurred during a period of economic downturn, the attacks in Spain happened when the world economy was experiencing growth. We believe that the difference in the impact can also be explained by examining the targets of the attacks. The 9/11 attacks happened in Manhattan, the financial center, while the bombings in Madrid were targeted at a transport system.

Further evidence of the impact of terrorism on financial markets is offered by some impact studies. Among existing literature, this paper is most closely related to that by Chen and Siembs (2004) which examines the US capital market reaction to 7 terrorist and 7 military attacks over the period 1915-2001, using an event-study approach. They apply their analysis to other capital markets as well, but focus on the impact of only two events: the 9/11 terrorist attacks and Iraq's invasion of Kuwait in 1990. They find that after these two events, US capital markets rebound and stabilize quicker than other markets, and that US markets are more resilient now than in the past, which they attribute to the strength of the banking and financial

[2] The major worldwide equity markets experienced sharp and rapid declines, demonstrating that market participants perceived the 9/11 event as a global shock. In contrast, the 2004 terrorist bombings in Madrid had much less effect on the financial markets. The Dow Jones EURO STOXX fell by about 3% on March 11, and continued to drop during the following days but had recovered almost completely by the end of the month. Similarly, after a small decline, the S&P 500 returned to pre-March 11 levels in less than a month (Johnston and Nedelescu (2005)).

sectors in the US. One of the main conclusions of their paper is that financial markets are efficient in absorbing the shocks caused by terrorist attacks and can continue to function in an effective way. Compared with the work by Chen and Siembs (2004), this study covers a much wider range of terrorist events (77 versus 7) and applies not only the traditional event-study approach, but more rigorous econometric techniques such as the non-parametric methodology and the filtered GARCH-Extreme Value Theory approach.

Eldor and Melnick (2004) study how stock and foreign exchange markets react to terrorism in Israel. The authors consider 639 terror attacks during the period from 1990 to 2003 and categorize the data by location, target, type of attack and number of casualties. They show empirically that terrorism has a permanent negative effect on stock markets but not on foreign currency markets. They conclude that these markets are efficient in incorporating news about terrorist attacks, and that there is no evidence that markets have become desensitized to terror over time.

Several studies consider the effects of the September 11 attacks exclusively on the stock market. Carter and Simkins (2001) examine the impact of this event on airline stock returns. They test whether market reaction on the first trading day after the attack is the same for each airline or, alternatively, whether it distinguishes among airlines based on company characteristics. They find that market differentiates among various airlines based on their ability to cover short-term obligations as measured by a ratio of cash and equivalents to total assets. According to their study, airlines with low liquidity are penalized the most. No statistical significance is found for company characteristics such as size, leverage and performance.

Other research focuses on the economic consequences and associated costs of terrorism. In their paper, Abadie and Gardeazabal (2003) study the effects of terrorism on economic activity. Karolyi and Martell (2006) analyze the long-term economic impact of terrorism. The authors examine the impact of terrorist attacks on the stock price of targeted companies. They find that the impact varies according to the domicile of the target company and the country in which the attack occurs. They conclude that in countries which are wealthier and more democratic, attacks are associated with greater share price reactions.

According to Raby (2003), airline, travel, tourism, accommodation, restaurant, postal and insurance industries are particularly susceptible to increased terrorism risks. Regions and economies where these industries are concentrated are likely to suffer most from falls in output and employment.

3. Terrorism risk

From an economic and financial standpoint, terrorism has been described as having several negative effects such as a reduction in the human and physical capital of a country, increased costs of financial and other counter-terrorism regulations, vulnerability of critical infrastructure (power plants, nuclear facilities, chemical factories, bridges, pipelines and water supply), increased financial instability, destruction of market infrastructure and operations and a decrease in investor confidence (see Johnston and Nedelescu (2005), Bonturi et al. (2002)). Because of enormous loss potential, terrorism risk may put high financial demands on insurance and reinsurance businesses and induce high insurance premiums. Today insurance companies mostly transfer this risk to reinsurance businesses. When dealing with terrorism risk, the main challenge for both types of financial institutions lies in its quantification. Even though some models have been proposed to handle this problem, existing approaches are linked with catastrophe modeling.

In many ways, terrorism risk is similar to the risk of natural hazards such as floods, earthquakes, hurricanes and storms. In all these events, there is enormous loss potential and these events can affect entire economies. For example, the 9/11 attacks have evidenced that terrorism is potentially a catastrophic risk. At the same time, there are several crucial differences between terrorist attacks and the above-mentioned extreme events. Unlike terrorist attacks, catastrophes are natural events that occur without intent and their conceivable place of occupance may be predicted with less difficulty. Terrorist events are characterized by dynamic uncertainty in terms of their type (suicide bombing, armed assault, kidnapping etc.), their target (military, personnel, government, facilities etc.) and place and time of occurrence. Terrorists may respond to security measures by shifting their attention to new targets or by changing the type of terrorist attack, the place or the time of its occurrence. In other words, they behave strategically. In contrast, the actions that can be taken to reduce the damage from possible natural disasters do not affect the probability and the place of occurrence of these events.

The main challenge is to predict the likelihood and the financial consequences of terrorist attacks and to quantify exposure to terrorism risk. In addition, when modeling this risk, analysts are faced with a limited availability of historical data on terrorism losses. But even if these data were easily accessible, it would not necessarily reflect the changing expectations of planned terrorist activities today. In contrast, probabilities and consequences of natural hazards can be modeled and quantified more easily using well-defined methods and historical data. Because of the above-mentioned characteristics, it is much more difficult to manage terrorism risk than the risk of natural hazards. This, in turn, calls for more studies on terrorism risk, and our work is the first that analyzes differences in the impact of terrorist attacks and natural disasters on the behavior of financial markets.

4. Empirical analysis

4.1. Research questions

When implementing our empirical study, we address the following research questions:

- *Research Question 1: Do terrorist attacks have a significant effect on global, European, American and Swiss stock markets?*
- *Research Question 2: Do terrorist attacks have a significant effect on such industry indices as insurance, travel, airline, oil and gas, financial and banking?*
- *Research Question 3: Do terrorist attacks have a significant effect on such industry indices as defense and pharmaceutical/biotechnology?*
- *Research Question 4: Do terrorist attacks have a significant effect on the commodity and gold markets?*
- *Research Question 5: Do terrorist attacks have a significant effect on the bond market?*
- *Research Question 6: Do terrorist attacks have a significant effect on financial markets on the event-day only, in the post-event window or both?*
- *Research Question 7: Do terrorist attacks have a significant effect on financial markets which is similar to that of natural catastrophes and financial crashes?*

4.2. Data

We use two types of data sets. The first data set includes daily prices of financial market indices (see Table 1 for a list of indices). We obtain this data from DataStream and, for each index, we consider available daily prices for the period from January 4, 1994 until September 16, 2005 (this corresponds to 3054 data points).[3] We compute the logarithmic daily percentage index

[3]For FTSE Global Banks and FTSE Global Financials, data are available from January 2, 1996; for MSCI Europe Insurance and MSCI Europe Airlines, from January 2, 1995; for FTSE Eurozone Bond Index from May 4,

returns using the identity of:

$$R_{i,t} = LN(P_{i,t}/P_{i,t-1}), \tag{1}$$

where $R_{i,t}$ is the return on the index for period t, $P_{i,t}$ is the price of the index at the end of period t, and $P_{i,t-1}$ is the price of the index at the end of period $t-1$.

The second data set includes information on terrorist events. We construct a database of terrorist events using publicly available information on terrorism (mainly provided by the Terrorism Research Center (TRA (2000)) and the UK Foreign and Commonwealth Office (FCO (2005))). As limited availability of historical information on terrorist events is often considered to be a restriction to modeling terrorism risk, we consider the data collection implemented in this study to be one of the first important steps in approaching the topic.

Data cover 77 terrorist events that occurred in 25 countries[4] from January 1994 to August 2005. Though our list is subjectively determined, we select those terrorist attacks that are mentioned as significant in the aforementioned sources. Each terrorist event is characterized by the date of attack, its type (armed assault, suicide bombing, bombing), the target, the place of occurrence and the number of people injured, killed or kidnapped.

When comparing the impact of terrorist attacks with the effect of other extreme events, we consider 4 financial crashes and 19 natural catastrophes that happened during the 11-year period considered. The financial crashes include the 1994 Mexican peso crisis, the 1997 mini-crash due to the Asian financial crisis, the 1998 Russian financial crisis and the 2001 Argentina crisis. The natural catastrophes include earthquakes, storms, floods, cyclones, typhoons, tornadoes, tsunami and hurricanes.

4.3. Methodology

In general, the event-study methodology is the most commonly used method to study the impact of events (see Brown and Warner (1980), Chen and Siembs (2004), Abadie and Gardeazabal (2003)). This methodology, however, imposes restrictive requirements on the behavior of indices' returns. In this paper, we go beyond this traditional tool and implement two other approaches. These are the non-parametric conditional distribution estimation approach and the filtered

1998; and for FTSE US Bond Index, from December 31, 1999. Regarding S&P 500 Index, data from September 12 to 19, 2001 were not available as the stock market was closed due to the 9/11 attacks.

[4]Argentina, Austria, Chile, Colombia, Egypt, France, Germany, Greece, India, Indonesia, Ireland, Israel, Japan, Malaysia, Mexico, Morocco, Pakistan, Peru, Philippines, Russia, Spain, Thailand, Turkey, the UK and the USA.

GARCH-EVT method. With the former, we gain the flexibility of having no assumptions about the parametric form of the data. With the latter, we account for the volatility background, possible dependence among returns and the fat tail nature of their distribution. Note that the GARCH-EVT approach is quite computationally intensive and we are able to check the impact on the event-day only.

4.3.1 The event-study approach

We use the event-study methodology to measure the magnitude of the effect of considered extreme events on the behavior of stock, bond and commodity markets. To examine whether an event has any impact on the market, we measure event-day abnormal returns (ARs) and cumulative abnormal returns (CARs) and test their statistical significance.[5] As the event-study approach is a well-known technique, we do not provide a more comprehensive overview of this methodology in our work.

4.3.2 Non-parametric conditional distribution approach

Non-parametric estimation is a statistical method that allows a functional form of a fit to data to be obtained without imposing any parametric assumptions. For example, a Kernel estimation of an economic model $y = M(x) + u$, requires no specification of a regression function $M(x) = E(y|x)$ and the distribution of error terms. This way, non-parametric estimation lets the data speak for itself and overcomes a disadvantage of parametric econometrics when inconsistency between data and a particular parametric specification would result in non-robustness. At the same time, this gain in flexibility of approach is not without costs, as non-parametric modeling has to deal with, for example, a selection of a bandwidth and a type of Kernel function. We do not intend to give a comprehensive overview of the fundamentals of non-parametric methodology. Rather, we want to describe an application of this powerful tool to study the impact of terrorism on different financial markets. We view this application, compared to event methodology, for example, as an alternative way of studying the impact. Note that when we analyze an impact of some event by implementing the event-study approach, we check the statistical significance of the effect of this event by means of some test statistics. The latter, in turn, imposes some restrictions, since test statistics require some distributional assumptions

[5]We apply a mean-adjusted return approach to compute ARs. CARs are computed over an interval of 6-days in the post-event window.

with respect to the abnormal returns or cumulative abnormal returns (CARs) which have to be satisfied.

We apply a local polynomial regression (LPR) to time series data to get a non-parametric conditional distribution of stock, bond and commodity index returns. We do not compute any test statistics to check the significance of negative abnormal and/or extreme movements in the market due to terrorism. Instead, for each index and terrorist event, we analyze the value of conditional probability of a return - which is less than or equal to the one empirically observed on the day of the event. The *abnormality* in the return corresponds to conditional probability in the interval $(0.05; 0.10]$. Where this probability is 5% or less, we interpret the return as *extreme*. This is our subjective approach to distinguishing between extreme and abnormal movements. We assume that a terrorist attack has an impact on the index if it leads to negative abnormal and/or extreme event-day returns. Since we are interested in knowing not only the immediate reaction of the market to the event, but also the market response over some interval of time in a post-event window, we estimate a non-parametric conditional distribution of non-overlapping 6-day CARs. We make our inference about the impact of terrorist attacks in the aftermath of the event by looking at 6-day CARs in a way similar to the one described for returns. Below we provide a description of the non-parametric estimation implemented in this paper.

Let us consider a conditional distribution function $\pi(z|x) \equiv P(Z_i \leq z|X_i = x)$. Since we work in the time series context, X_i is a vector of lagged values of Z_i that are returns on an index. If we assume $Y_i = I(Z_i \leq z)$ then $E(Y_i \mid X_i = x) = \pi(z|x)$, consequently, the problem of estimation may be viewed as regression of Y_i on X_i.

Keeping this in mind and applying a local polynomial fitting to our time series data of index returns R_i, we minimize the following expression:

$$\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1(X_i - x_0))^2 K_h(X_i - x_0), \tag{2}$$

where $Y_i = \mathrm{I}(R_i \leq r_t)$ with r_t standing for empirically observed (realization) return on the day of terrorist attack t, $i = (1, \ldots, n)$, n is a sample size and n=200; $X_i = R_{i-1}$, $x_0 = r_{t-1}$; h is a bandwidth; K_h is a Kernel function.

Figure 1 provides a graphical illustration of the idea of non-parametric estimation implemented in this work. We take the 9/11 attacks as an example and build the conditional cumulative distribution function of returns on FTSE All World when conditioning is done on the return on September 10, 2001, a day before the 9/11 attacks. We find the conditional cumulative probability of the return on FTSE All World, which is less than or equal to that on September

11, 2001 to be 0.037. Since this value is less than 0.05, we conclude that this terrorist event has an extreme event-day effect on this index.

We use a normal reference bandwidth selector (Fan and Yao (2003)), which defines an optimal bandwidth $\hat{h}_{opt,n}$ for the Epanechnikov kernel as $2.34\sigma_s n^{-1/5}$, where σ_s is a standard deviation of a sample. Implementation of this model leads to point estimates $\widehat{\beta_0}$ and $\widehat{\beta_1}$:

$$\widehat{\beta} = (\mathbf{X}'W\mathbf{X})^{-1}\mathbf{X}'WY, \tag{3}$$

where W is the diagonal matrix whose i^{th} element is $K_h(X_i - x_0)$, and $\mathbf{X}$ is a design matrix with a first column of ones. Obtained this way, a point estimate $\widehat{\beta_0}$ corresponds to a conditional probability of return on index, which is less than or equal to that empirically observed on the day of the event (a terrorist attack) and when conditioning is done on the value of return on the previous day. The same logic applies to a sample of 200 non-overlapping 6-day CARs. We compute the value of the CAR on which conditioning is done as

$$CAR_{t-1} = \sum_{j=-1}^{j=-6} AR_j, \tag{4}$$

where AR_j is the abnormal return on an index at time j. We also implement a non-parametric estimation when conditioning is implemented on the average of the returns $\bar{R}$. We believe that this approach improves the inference since the average return reflects normal market conditions better than just one return on the day before the attack.

4.3.3 GARCH filter with an extreme value theory approach

When studying the impact of extreme events on financial market behavior, one can compare the event-day return on the index with the value at risk (VaR) predicted for this day and computed for different levels of significance. In the case of terrorist attacks, if the return on the day of a terrorist event is lower than the computed value of VAR, we may conclude that a considered terrorist attack had an impact on the index. This method of studying the impact of events relates to the tail estimation of financial time series and requires a well-chosen way to compute the VaR that from a statistical point of view has a good predictive performance (a good fit model). In their recent paper, Kuester et al. (2006) give an extensive and detailed overview and comparison of alternative strategies to predict VaR. They implement their study using the NASDAQ Composite Index and show that the hybrid method that combines a heavy-tailed generalized autoregressive conditionally heteroscedastic (GARCH) filter with an extreme value theory (EVT) approach performs better than other methods. More about the VaR measurement, using GARCH and EVT theory can be found in the papers by Christoffersen et al. (2001), Longin (2005) and Bali et al. (2008).

The GARCH method works as follows. First, we apply a time-varying volatility model to the time series of returns. We assume the following dynamics of returns:

$$X_t = \mu_t + \sigma_t Z_t, \tag{5}$$

where X_t is a strictly stationary time series representing daily observations of negative log returns on index, and innovations Z_t are white noise process and have a marginal distribution function $F_Z(z)$. We assume the Gaussian distribution for innovations.

We assume that μ_t and σ_t are measurable with respect to $\Im_{t-1}$, the information about the return process available up to time $t-1$. Similar to the paper by McNeil and Frey (2000), we use the parsimonious but effective AR(1) model for the dynamics of the conditional mean:

$$\mu_t = \varphi X_{t-1} \tag{6}$$

and GARCH(1,1) process for the conditional volatility:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{7}$$

where $\alpha_0 > 0$, $\alpha_1 > 0$ and $\beta > 0$, $\epsilon_t = X_t - \mu_t$ and $\alpha_1 + \beta < 1$.

For each terrorist attack, we take a sample from 200 to 2500 past return observations[6] (starting one day before the attack) and fit this model to the data by means of the pseudo-maximum likelihood method to get the estimates of parameters $\widehat{\theta} = (\widehat{\varphi}, \widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta})$. Estimates of the conditional mean series $(\widehat{\mu}_{t-n+1}, ..., \widehat{\mu}_t)$ and the conditional standard deviation series $(\widehat{\sigma}_{t-n+1}, ..., \widehat{\sigma}_t)$ are obtained recursively from (9) and (10) using reasonable starting values. When correctly specified and with a good fit, this model allows us to obtain filtered residuals

$$(z_{t-n+1}, ..., z_t) = \left(\frac{x_{t-n+1} - \widehat{\mu}_{t-n+1}}{\widehat{\sigma}_{t-n+1}}, ..., \frac{x_t - \widehat{\mu}_t}{\widehat{\sigma}_t}\right) \tag{8}$$

that are approximately iid, which is an important requirement for the EVT approach applied below. Finally, the estimates of the conditional mean and variance for day $t+1$ are given by $\widehat{\mu}_{t+1} = \widehat{\varphi} x_t$ and $\widehat{\sigma}_{t+1}^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 \widehat{\epsilon}_t^2 + \widehat{\beta} \widehat{\sigma}_t^2$, where $\widehat{\epsilon}_t = x_t - \widehat{\mu}_t$.

We estimate the tail of the standardized residuals by means of the EVT, namely, by applying the Peak-Over-Threshold methodology (POT). The latter approach focuses on the distribution

[6] We vary the sample size to get good backtesting results.

of excess returns over some threshold and applies a key result that the Generalized Pareto Distribution (GPD) is the limit distribution of scaled excesses of iid random variables over high threshold. This distribution has the following cdf for $\xi \neq 0$:

$$H_{\xi,\beta}(y) = 1 - \Big[1 + \frac{\xi y}{\beta}\Big]^{-1/\xi}, \tag{9}$$

where $\beta > 0$, $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\beta/\xi$ and when $\xi < 0$. When $\xi = 0$, the expression in (12) gets the form of $H_{\xi,\beta}(y) = 1 - \exp\Big(\frac{-y}{\beta}\Big)$. When implementing the EVT estimation, we first order the residuals $z_{(1)}, ..., z_{(n)}$ and fit the distribution in (12) to the data $(z_{(1)} - z_{(k+1)}, ..., z_{(k)} - z_{(k+1)})$, the excess amounts over the threshold $z_{(k+1)}$ with k standing for the number of data in the tail. The quantile estimate $\widehat{z}_q$ for $q > 1 - k/n$ is

$$\widehat{z}_q = z_{(k+1)} + \frac{\widehat{\beta}_k}{\widehat{\xi}_k}\Big(\Big(\frac{1-q}{k/n}\Big)^{-\widehat{\xi}_k} - 1\Big). \tag{10}$$

Finally, we compute the estimate of the VaR. If we denote the marginal distribution of X_t as $F_X(x)$ and let $F_{X_{t+1}+...+X_{t+k}|\Im_t(x)}$ be the predictive distribution of returns over the next k days, then the quantile of the latter distribution is given by

$$VaR_q^t = x_q^t(k) = \inf\{x \in R : F_{X_{t+1}+...+X_{t+k}|\Im_t(x)} \geq q\}. \tag{11}$$

Because $F_{X_{t+1}|\Im}(x) = P\{\sigma_{t+1}Z_{t+1} + \mu_{t+1} \leq x \mid \Im_t\} = F_Z((x - \mu_{t+1})/\sigma_{t+1})$ we can compute VaR as

$$\widehat{VaR}_q^t = \widehat{x}_q^t = \widehat{\mu}_{t+1} + \widehat{\sigma}_{t+1}\widehat{z}_q, \tag{12}$$

where $\widehat{z}_q$ is the upper qth quantile of the marginal distribution of Z_t obtained using (13). Computed this way, the VaR accounts for the volatility background and fat-tail nature of the distribution of index returns. These are two important stylized facts of most financial return series.

To evaluate the predictive power of the above approach, we implement a backtesting procedure described in McNeil and Frey (2000). We update the AR(1)-GARCH(1,1) model parameters for 500 moving windows and produce 500 one-step-ahead forecasts of $\widehat{VaR}_q^t = \widehat{x}_q^t$ that are subsequently compared with observed values of returns x_{t+1} for $q \in \{0.90, 0.95, 0.99\}$. We implement this procedure when studying the impact of every terrorist attack. A violation is said to occur when $x_{t+1} > \widehat{x}_q^t$. Finally, given that the total number of violations is binomially distributed, we test the hypothesis that the model estimates the conditional quantiles correctly.

4.4. *Empirical results*

4.4.1 Summary

Our empirical study provides the following answers to the research questions addressed in this work (see Table 1 for a summary of the results):

- *Research Question 1: Do terrorist attacks have a significant effect on global, European, American and Swiss stock markets?*

  Yes they do. The results obtained show a significant negative impact of terrorist events on the above mentioned markets: according to the event-study, 55 out of 77 terrorist attacks (56 in the non-parametric case, 45 according to the GARCH-EVT method) have a significant negative impact on the behavior of at least one of these markets. The Swiss market is affected by the highest number of attacks while the American market is affected by the lowest number of events. The reasons for a strong reaction by the Swiss market to terrorist events can relate to several factors. This may be because the SMI index is comprised of fewer companies than the S&P 500 (less broad index). In addition, the SMI's sensitivity may be explained by the fact that this index includes stocks of companies that operate internationally and are potentially more sensitive to extreme events due to the nature of their business. The results obtained for the S&P 500 are quite reasonable given the fact that only 4 out of 77 terrorist attacks we consider took place in the US. In addition, the resilience of the American market to terrorist attacks can be explained by the stable banking/financial sector in the US at the time of the events, which provided adequate liquidity to promote market stability.

- *Research Question 2: Do terrorist attacks have a significant effect on such industry indices as insurance, travel, airline, oil and gas, financial and banking?*

  Yes they do. The empirical evidence suggests a significant negative impact of terrorist events on the above-mentioned industries. According to the event-study 61 terrorist attacks (55 in the non-parametric case, 41 according to the GARCH-EVT method) lead to significant negative return movements in at least one industry index. Insurance and airline sectors exhibit the highest susceptibility to these events (the MSCI Europe Insurance is affected by the highest number of attacks), while the banking sector is least affected. These results are quite intuitive. Terrorist attacks often lead to fatalities and significant damage to property which explains a high sensitivity of the insurance sector to terrorist risk. The results support the conclusions of several studies (see Raby (2003), Bonturi et al. (2002), Abadie and Gardeazabal (2003), Enders et al. (1992)) that identify the airline, travel, tourism and insurance sectors as those which are particularly sensitive to terrorist events. With respect to the lower level of impact on the banking sector, it is possible that banks' operations are not directly related to the businesses that suffered from the terrorist events. Finally, the oil/gas industry shows both significant negative and positive return response. We observe this effect at both global and European levels. We observe a negative reaction of these indices more often than a positive, and we can explain this reaction as a fear of possible economic slowdown and a decrease in consumer confidence. This especially relates to the transportation sector, for example, by a drop in air travel. In turn, this leads to a lower oil demand and a decrease in oil prices. At the same time, a positive effect on oil prices is often related to the place of the attack (whether it can cause a danger to oil production and transportation) and the oil market conditions at the time of the event (if an attack occurs when the market is tight because of increasing global demand). Importantly, in this study we analyze the impact of terrorist events on returns on the event-day and in the post-event window of 6 days after the attacks. Therefore, conclusions drawn from our investigation relate to the markets' short-term reaction only.

- *Research Question 3: Do terrorist attacks have a significant effect on such industry indices as defense and pharmaceutical/biotechnology?*

  Yes they do. The analysis of the impact shows both positive and negative reactions of these indices across all methodologies. The pharma/biotech index is affected less often in a negative way compared to the defense index. The former index also shows a significant positive response to more terrorist attacks. Among the events that negatively affect at least one of these indices are the 2002 bombing in Peru, the 2003 suicide bombings in Israel and the 2004 bombings in Russia.

  In terms of a positive response, we observe a significant positive impact on both sectors for the 1996 suicide bombing in Israel and the 2002 bombings in Pakistan. We find a significant positive impact on the pharma/biotech industry for the 2002 bombings in Indonesia/Bali and the 2005 armed assault in Colombia. We identify a significant positive impact on the defense industry for the 1995 bombing in Oklahoma City. Finally, when we examine the post-event impact over a longer time window (11-day CARs and 30-day CARs), we find a significant positive response of both indices to such terrorist attacks as that of 9/11, the bombings in Madrid and Egypt in 2004 and in London in 2005. One of the explanations of the positive impact on the defense and pharma/biotec indices that we may suggest is that terrorist events may induce an increase in government expenditures on defense and on research in the pharma/biotech area in relation to preventive actions against possible chemical or biological terrorist attacks.[7]

- *Research Question 4: Do terrorist attacks have a significant effect on the commodity and gold markets?*

  Yes they do. The analysis of the impact shows both significant positive and negative reactions of the commodity and gold market returns to terrorist events. The latter market shows more negative sensitivity to terrorist events compared to the commodity and bond markets. Given that gold is usually considered to be a 'safe-haven' asset, these empirical results remain difficult to explain. Commodity and gold markets respond positively to some terrorist events (the 9/11 attacks) and show no significant reaction to others (the bombings in Egypt in 2004). Finally, some events, the 2005 bombings in London for example, cause significant negative return movements in the commodity index and have no effect on the gold index. Such behavior implies that investing in the commodity/gold markets as a hedging strategy against terrorism risk may not always work because, with terrorist events, these markets can react negatively.

- *Research Question 5: Do terrorist attacks have a significant effect on the bond market?*

  Yes they do. The analysis of the impact shows both significant positive and negative reactions of the bond market returns to terrorist events. We observe the negative impact of some attacks mostly on the event-day only. Compared to other bond indices, the Global Government Bond Index experiences significant positive return movements more often than negative return movements. The FTSE US Government Bond Index displays the lowest level of impact, both positive and negative. As with commodities and gold, investing in bonds can be a possible hedging strategy against terrorism risk. However, one should be aware of the possibility of a significant negative response of these assets' returns to terrorist attacks.

- *Research Question 6: Do terrorist attacks have a significant effect on financial markets*

[7] The OECD report also suggests the possibility of a positive reaction by the defense industry to terrorist events (Bonturi et al. (2002)).

on the event-day only, in the post-event window or both?

The empirical results show that terrorist events lead to a significant response in financial market returns in all the above-mentioned cases. In most cases, the event-day stock return movements associated with attacks are extreme and the strength of the impact declines in the post-event period.

Regarding commodity markets, significant return movements in the gold index, both negative and positive, are extreme and often observed in the post-event period. In contrast, we find a negative reaction by the Goldman Sachs Commodity Index in all periods, and we observe a positive response mostly on the event-day.

Among bond indices, the global and European bond markets react negatively in all periods, while return movements are more often extreme than abnormal. Unlike these two markets, the US bond market responds positively to terrorist events mostly on the event-day and associated returns have an abnormal character.

All financial markets perceive terrorist attacks as unusual events. While some of them see the effects of these events as occurring mostly on the event-day only (the US bond market), some markets take a longer time to evaluate the impact and reveal their reaction mostly in the post-event period (the gold market). Finally, some markets (stocks, commodities, global and European bonds) react to terrorism either on the event-day or in the post-event window or both.

- *Research Question 7: Do terrorist attacks have a significant effect on financial markets which is similar to that of natural catastrophes and financial crashes?*

  There are both similarities and differences between the impact of terrorist events on financial markets and the effect of financial crashes[8] and natural disasters.

  While the European and Swiss markets show high susceptibility to terrorist attacks and natural catastrophes, their response to financial crashes is less negative. At the industry level, the insurance and airline sectors show a negative sensitivity to all types of the extreme events. Financial crashes demonstrate a strong negative impact on the banking and financial sectors. This is in contrast to the effect of terrorist attacks and natural

[8] A recent paper by Wang et al. (2009) applies the event-study methodology and multivariate regression analysis to study how a stock market crash affects individual stocks and if stocks with different financial characteristics are affected differently.

catastrophes that do not cause a strong negative response in the sectors mentioned above. Similar to terrorist events, we observe both a positive and a negative impact of natural disasters and financial crashes on such industries as oil/gas and pharma/biotech.

The event-day negative returns associated with financial crashes and terrorist events are extreme. The sensitivity of stock markets to these events declines in the post-event window. In contrast, natural disasters are associated with extreme return movements more in the days following the events. This result may be because markets need more time to evaluate the long-term consequences of natural disasters on returns compared to the two other types of events.

Both terrorist events and natural disasters cause positive and negative return movements in the commodity/gold and bond markets. Among the latter markets, the US bond market shows the least impact from all extreme events considered. With respect to the impact of financial crashes, our empirical findings confirm a traditional perception of the commodity and bond markets as those providing 'safe-haven' investment opportunities in times of crises. This is because these markets react positively to financial crashes.

4.4.2 Empirical results across different methodologies

4.4.2.1. The event-study approach

Analysis of the response of four stock indices - FTSE All World, MSCI Europe, S&P 500 and SMI to terrorist attacks shows that 22 out of 77 attacks have no impact on any of these stock markets. Among these events are not only local attacks that are characterized by very little or no damage to property and people as for example, the bombing in Israel on May 27, 2001, but also such events as the attacks in Argentina on July 18, 1994 that are considered to be one of the worst in terms of fatalities. The bombings in Sri Lanka/Colombo on July 24, 2001 are among the worst in terms of insured property loss during the period 1970-2001. In other words, the impact of attacks on stock markets is not necessarily in direct relation to their magnitude in terms of insured losses and fatalities.

Our investigation shows that 55 out of 77 terrorist events have a significant negative impact on at least one stock index. FTSE All World and Swiss indices are affected by the highest number of events while the S&P 500 Index is affected by the least number of attacks. The 9/11 attacks as well as the suicide bombing in Israel on June 19, 2002 and bombings in Madrid 2004, in Egypt 2004 and in the UK 2005 are good examples of events that have a negative impact on stock markets at both global and local levels (see Table 2). We find that the 9/11 terrorist attacks have a significant negative effect on global, European, American and Swiss stock markets both on the event-day and in the post-event window. The S&P 500 shows the strongest negative reaction in the post-event window compared to other indices, which reflects a prolonged negative effect of the 9/11 event on the American market. As to the negative impact of this event on the European market, our results find support in the empirical paper by Chen and Siembs (2004), where the authors conclude that European capital markets experience significant negative 6-day CARs due to the 9/11 attacks.

The empirical results for various industry indices show that 62 out of 77 terrorist events have a significant negative effect on at least one of them. The insurance sector is affected by the highest number of events while the banking and oil/gas industries are affected by the lowest number of attacks. Within the insurance sector, MSCI Europe Insurance experiences the most negative impact and is closely followed by FTSE All World Non-Life Insurance. These results are quite intuitive since terrorist attacks often lead to fatalities and significant damage to property that explains a high sensitivity of the insurance sector to terrorism risk. Unlike the insurance industry, the banking sector is affected by the least number of attacks. It is possible that banking operations are not directly related to the businesses that suffered from the terrorist events.

We find evidence of the significant negative impact of terrorist attacks on the FTSE All World Travel and MSCI Europe Airlines. For both, almost half of the attacks considered lead to a significant negative reaction. While the FTSE All World Travel is affected more often in the post-event period, we observe this type of impact for MSCI Europe Airlines less frequently. In addition, when characterizing the impact on the airline index, we see that more than half of the terrorist events (out of 31 that have an impact) cause significant negative return movements on the event-day and in the post-event window, reflecting a high susceptibility of this sector to terrorism risk. These results support conclusions of several studies (see Raby (2003), Bonturi et al. (2002), Abadie and Gardeazabal (2003), Enders et al. (1992)) that identify airline, travel, tourism and insurance sectors as those that are particularly sensitive to terrorist events.

Analysis of the ARs and 6-day CARs shows evidence of significant negative as well as positive impact of terrorist attacks on the aero/defense and pharma/biotech industries. The pharma/biotech index is affected less often in a negative way compared to the aero/defense index. It also shows a significant positive response to more terrorist attacks.

The aero/defense and pharma/biotech sectors are not the only industries that experienced both a positive and a negative impact of terrorist attacks. An analysis of the reaction of the oil/gas sector also reveals these two types of impact that is observed at both global and European levels. Some terrorist events first cause a significant negative event-day response in the oil/gas industry and then lead to a significant positive impact in the post-event period. This behavior is observed, for example, for the 2004 bombings in Madrid (FTSE Europe Oil/Gas). In this case, we may possibly explain the immediate negative reaction of the market by the fact that these bombings are targeted at the transportation system. As new information is processed with respect to the long-term effect, the market reveals no fear.

4.4.2.2. The non-parametric approach

We obtain the results in relation to the event-day impact of attacks when conditioning is implemented on the average return.[9] Similar to the findings of the event-study approach, the impact of attacks on the stock markets is not necessarily in direct relation to their magnitude in terms of insured losses and fatalities. Our investigation shows that 56 out of 77 terrorist events have an impact on at least one stock market under consideration. The Swiss and European indices are affected by the highest number of events, while the S&P 500 Index is affected by the least number of attacks. These results are similar to those revealed in the event-study.

For FTSE All World, MSCI Europe, S&P 500 and SMI, terrorist attacks more often lead to an event-day negative response and less often to a prolonged negative reaction. Similar to the results of the event-study approach, all stock indices experience extreme event-day negative return movements more often than abnormal return movements. The strength of the impact declines in the post-event period. This result also applies to the industry indices with the exception of airline, aero/defense and pharma/biotech sectors. Similar to the findings of the event-study approach, the negative impact of terrorist events is found at both global and local levels (see Table 3).

We find that 55 terrorist attacks have significant negative impact on at least one industry index. This result reflects the more conservative nature of the non-parametric approach

[9]There are several reasons for this. First, as we mentioned before, the return on the day before the attack might not represent normal market conditions as accurately as the average return. Secondly, analysis of the data shows that there are quite a few terrorist attacks that have low event-day impact when conditioning is implemented on the previous return, no impact in the post-event window and no event-day impact when conditioning is performed on average returns. In addition, these attacks are such that their magnitude or place of occurrence suggest that no impact on the given stock index is a reasonable result.

compared to the event-study methodology. The latter suggests that a larger number of events cause a negative market response. This result can be due to restrictive assumptions imposed by test statistics used in the event studies. At the same time, the findings across different industries are quite similar among these methodologies. The insurance and airline sectors show high sensitivity to terrorism, while the FTSE Europe Oil/Gas, followed by the banking sector is affected by the lowest number of attacks. Similar to the findings of the event-study approach, the aero/defense, pharma/biotech and oil/gas sectors exhibit both positive and negative abnormal return movements associated with terrorist events. Events that cause positive reaction are similar to those identified in the event-study (see Tables 6-7).

4.4.2.3. The GARCH filter with EVT approach

In contrast to the other two approaches, the results provided by this method describe only the event-day impact of terrorist attacks.[10] 45 out of 77 terrorist events have a significant negative impact on at least one stock index. More than half of these events cause extreme rather than abnormal event-day returns. The Swiss stock index is most often affected. These results are similar to those revealed in the event-study and in the non-parametric approach. At the same time, American and European markets exhibit the lowest level of event-day impact.

At the industry level, 41 out of 77 attacks lead to significant negative event-day return movements in at least one industry index. Similar to the findings of the other two methods, the insurance and airline industries exhibit the highest susceptibility to terrorism (the MSCI Europe Insurance is affected by the highest number of attacks). The oil and pharma/biotech sectors exhibit the least negative event-day impact. The GARCH-EVT method suggests a greater negative impact on the aero/defense index and a lesser negative impact on the pharma/biotech index compared to the other two methods. At the same time, it also displays the presence of significant positive event-day return movements in these two sectors, a result which is similar to the findings of the other two methods.

We identify some terrorist events, namely the 1996 bombing in Sri-Lanka/Colombo, the February 1997 armed assault in the US and the February 1997 kidnapping in Indonesia as those causing a significant positive impact on the pharma/biotech sector. Other methods, however, do not identify the impact of these events. With the exception of the above-mentioned events, the list of attacks that lead to the positive event-day impact is similar across all methodologies (see Tables 6-7). Finally, concerning the oil/gas sector, the GARCH-EVT approach shows a

[10]See section 4.5 that explains the limitations of the GARCH-EVT method.

significant negative event-day impact of a smaller number of attacks and a significant positive impact of a greater number of attacks compared to the other two methods.

4.5. *Which method to use?*

Tables 1-4-5-6-7 summarize the findings of our empirical work across different methodologies. Comparing methodologies, the GARCH-EVT approach shows the least number of extreme events that lead to significant negative event-day return movements in the indices considered. Regarding positive impact, the event-study approach often reveals a significant effect of a lower number of events. For indices that experience both types of impact, the GARCH-EVT method shows more positive impact compared to other methodologies. The differences in impact across methodologies can relate to the underlying assumptions they impose on the market returns. The GARCH-EVT approach, for example, accounts for the volatility background, dependence and the fat-tail nature of the market returns. These are important characteristics of financial market returns that are not captured by the other two methodologies. However, the GARCH-EVT approach allows us to study the event-day effect only and is computationally intensive. These features are the significant drawbacks of the GARCH-EVT approach, and we believe, therefore, this approach is inferior to the other two methods. The event-study approach, on the other hand, is too simplistic since it imposes strong parametric assumptions which may not hold in reality. In contrast, for a non-parametric approach those parametric restrictions do not apply and it allows us to analyze post-event effects. Moreover, this approach requires less computational work than GARCH-EVT. Non-parametric estimation lets the data speak for itself and overcomes a disadvantage of parametric econometrics when inconsistency between data and a particular parametric specification would result in non-robustness. Therefore, we consider a non-parametric approach to be the most appropriate method among the three for analyzing the impact of terrorism on financial markets. In the following sections we show the robustness of this method and analyze how the results of this approach can be used for investors' portfolio diversification strategies against terrorism risk.

4.6. *An application: Portfolio diversification strategies*

In this section we analyze how Swiss and European investors can use the results of the non-parametric approach in their portfolio diversification strategies. We focus on Swiss and European investors because sectoral indices which are going to be used for hedging purposes against terrorism risk are mainly dominated by US stocks. Therefore, the impact of the hedging strategies on portfolio performance will be more pronounced for Swiss and European investors than for US investors due to the high correlation between hedging indices and the S&P 500 index.

First, we consider three Swiss investors who construct equity-only efficient portfolios one week before the September 11 terrorist attacks. Then we check the performance of these portfolios on the event-day, i.e. September 11, 2001, as well as one, two and three weeks after the attacks. The first investor holds only the Swiss market portfolio (i.e. the SMI). The second and third investors use the results of our non-parametric approach. In addition to the SMI, they include in their portfolios sectoral indices that react negatively[11] and positively[12]to terrorist attacks. The only difference between the portfolios of the last two investors comes from the structure of the covariance matrices of returns. The second investor applies an average-correlation technique (see Elton et al. (1978)) to estimate the covariance matrix of the asset returns based on historical data. She estimates the variances and a single average correlation of the returns and then constructs a covariance matrix with these values. To some extent, this investor constructs an international portfolio that includes indices affected by terrorist attacks. However, she doesn't take into consideration a possible direction of the impact of terrorist events (positive or negative). In contrast, the third investor incorporates a possible direction of the impact of these events. She calibrates a negative correlation to the sectoral indices which react inversely to the terrorist attacks and a positive average correlation to the rest of the covariance matrix.

The estimation of a covariance matrix of the returns explained above is quite simplistic. However, the results of DeMiguel et al. (2009) show that an investor could be better off ignoring data on asset returns and using the naive portfolio weights of $1/N$. The authors claim that *"there are still many miles to go before the gains promised by optimal portfolio choice can actually be realized out of sample."* As it is not the main focus of this study, we leave it to other researchers to develop better estimation schemes.

The out-of-sample performances of the portfolios considered in our analysis are presented in Table 8 via portfolio Sharpe Ratios.[13] On the event-day (i.e. September 11, 2009) all portfolios perform poorly, but the relative performance of the second investor is better than that of the

[11]Insurance, airline and travel indices.

[12]Aero/defense and pharma/biotech indices.

[13]A recent paper by Zakamouline and Koekebakker (2009) presents how generalised Sharpe Ratios can be used in the portfolio performance evaluation. See also Farinelli et al. (2008) and Jha et al. (2009) for other measures than the Sharpe Ratio in order to analyze portfolio performances.

first one, and the relative performance of the third investor is better than that of the second investor.[14] When the performance horizon increases, we see a similar pattern where the third investor outperforms the other two.

We follow the same steps when performing our analysis for European investors. We construct three different portfolios one week before the September 11 terrorist attacks and then evaluate their performance on the event-day as well as one, two and three weeks after the attacks. The first investor holds only the European market portfolio (i.e. the MSCI Europe index). The second investor holds the MSCI Europe index and incorporates the results of our study by investing in the indices that are affected by terrorist attacks. However, she does not account for the direction of the impact of these events. Finally, the third investor holds the MSCI Europe index and indices which are sensitive to terrorist attacks. In contrast to the second investor, she calibrates the covariance matrix in such a way that she imposes a negative correlation on the industries that react inversely to the terrorist attacks and a positive average correlation on the rest of the covariance matrix.

The out-of-sample performances of these portfolios are presented in Table 9. Similar to the results for the portfolios of Swiss investors, all portfolios of European investors perform poorly on the event-day. In relative terms, however, the third investor outperforms the second investor and the second investor outperforms the first investor[15] on the event-day and on the other days of analysis.

The overall results across portfolio diversification strategies of Swiss and European investors described above show that portfolios which account for a possible impact of terrorist events demonstrate better performance than those which ignore it. Moreover, the hedged portfolios of the Swiss investors outperform those of European investors, as the impact of the 9/11 attacks were worse for the Swiss market than for European markets (see Table 3). The findings of our study are useful as they reveal not only the indices which may be affected by terrorist events, but also the direction of the impact (positive or negative). In other words, an investor who uses the results of our paper and constructs an investment portfolio in such a way that she imposes a negative correlation on the industries that react inversely to terrorist attacks and a positive

[14]Although negative Sharpe Ratios could be misleading, it is not the case in our study. This is because during the performance evaluation period the standard deviation of the first portfolio is larger than that of the second, and the standard deviation of the second portfolio is larger than that of the third.

[15]Negativity of the Sharpe Ratios does not lead to a wrong conclusion in this case either. This is because during a performance evaluation period, the standard deviation of the first portfolio is larger than that of the second, and the standard deviation of the second portfolio is larger than that of the third.

average correlation on the rest of the covariance matrix, would outperform other investors who don't consider these inverse reactions.

4.7. *Robustness of the non-parametric methodology*

In order to show the robustness of the non-parametric approach, we continue our analysis with the SMI and MSCI Europe index returns and implement controls for interest rates, equity market integration, lagged spillovers and contemporaneous effects.[16] To do so, we run the following regressions

$$(R_{t+1}^{SMI} - Rf_{t+1}^{Swiss}) = \alpha + \beta_{\mathbf{1}}(R_{t+1}^{FTSE,World} - Rf_{t+1}^{World}) + \beta_{\mathbf{2}}(R_t^{SMI} - Rf_t^{Swiss}) + \beta_{\mathbf{3}}(R_{t+1}^{S\&P500} - Rf_{t+1}^{US}) + \epsilon, \tag{13}$$

$$(R_{t+1}^{MSCI,EU} - Rf_{t+1}^{EU}) = \alpha + \beta_{\mathbf{1}}(R_{t+1}^{FTSE,World} - Rf_{t+1}^{World}) + \beta_{\mathbf{2}}(R_t^{MSCI,EU} - Rf_t^{EU}) + \beta_{\mathbf{3}}(R_{t+1}^{S\&P500} - Rf_{t+1}^{US}) + \epsilon, \tag{14}$$

where R_{t+1} stands for the log-return on a day $t+1$ and Rf is a risk-free rate. In other words, we regress excess index returns on a constant, excess world equity index returns, lagged excess index returns and excess S&P 500 index returns which control for equity market integration, lagged spillovers and contemporaneous effects respectively.

Our aim is to compare the findings of a non-parametric approach when applied to the residuals of this regression analysis with the results that are obtained before the controls for different effects are introduced. Tables 10-11 show that the results are quite similar for the robustified and the unrobustified index returns. That is, for the SMI, 7 out of 10 events have an event-day impact using both robustified and unrobustified methods. For the MSCI Europe index, the event-day results are similar for 8 out of 10 events. In terms of the post-event window effects, 5 out of 6 events demonstrate similar results for both indices. Therefore we conclude that for the data under consideration, our non-parametric methodology is robust with respect to interest rates, equity market integration, spillover and contemporaneous effects.

5. Conclusions

This study shows the results regarding the global, regional, national and industrial effects of terrorist events on stock markets as well as the impact of attacks on commodities and bonds. Furthermore, it compares the impact of terrorist events on financial markets with the effect of

[16]Note that due to data restrictions we are able to perform the robustness checks from the beginning of 2000 until the end of 2005.

natural catastrophes and financial crashes. To do so, we present three different methodologies namely, event-study, non-parametric and filtered GARCH-EVT approaches. The results of our analysis show that a non-parametric approach is the most appropriate method among the three for analyzing the impact of terrorism on financial markets. It allows us to study both event-day and post-event window effects, it does not impose strong parametric restrictions, and it is not computationally intensive. We demonstrate the robustness of this method when interest rates, equity market integration, spillover and contemporaneous effects are controlled. Finally, we show how the results of this approach can be used for investors' portfolio diversification strategies against terrorism risk.

Approximately two-thirds of the terrorist attacks considered lead to significant negative impact on at least one stock market under consideration. The Swiss stock market is affected by the highest number of attacks, the American stock market by the lowest. The airline industry and insurance sector exhibit the highest susceptibility to terrorism, while the banking industry is the least sensitive. This is in contrast to financial crashes which demonstrate a strong negative impact on the banking sector. The analysis of the impact on the aero/defense, pharma/biotech and oil/gas sectors shows both a positive and a negative reaction. These indices behave similarly in case of the natural disasters and financial crashes.

The results of our study suggest several diversification strategies against terrorism risk. If concerned about this risk, investors should hold assets that can react positively to terrorist attacks or, alternatively, assets that have little or no negative sensitivity to this risk. In the first case, the US Government bond index is the safest choice followed by such industry stocks as aero/defense and pharma/biotech. However, given that these indices can also exhibit a negative response, investing in these industries as a diversification strategy against terrorist attacks may not always work. In the second case, a banking stock index can be a good investment. Note that, though this stock index is less sensitive to terrorist attacks, it exhibits significant negative return movements associated with financial crashes.

Regarding the other financial markets, investing in commodities is preferable to investing in gold as the gold market reacts more often negatively than positively. In addition, compared to the commodity market in general, the negative impact on the gold market is more long-lasting. At the same time, the commodity market also shows a short-term negative reaction to some terrorist events. This implies that investing in the gold and commodity markets may not always provide a good hedge.

A possible way to reduce negative exposure to terrorist events would be to avoid investing in the insurance, travel and airline stock markets or to short these indices. Note that the insurance and airline industries shows high negative sensitivity not only to terrorist attacks but also to financial crashes and natural disasters. This implies that by taking long positions in these stocks, investors may end up increasing the risk of losses in these cases where extreme events occur.

There are both similarities and differences between the impact of terrorist events on financial markets and the effect of other extreme events. For example, the insurance and airline industries show high sensitivity to all three categories of extreme events. The banking industry shows little negative impact of natural hazards, which is similar to the impact of terrorist attacks and in contrast to the effect of financial crashes. Terrorist attacks and natural disasters cause both positive and negative significant return movements in the commodity and bond markets. In contrast, financial crashes have a positive effect on these markets. Terrorist attacks and financial crashes cause an event-day return movement that has an extreme nature in general, with the strength of the impact declining in the post-event period. As for natural catastrophes, the negative impact is more often observed in the post-event period, implying that markets need more time to evaluate the long-term effect of these extreme events.

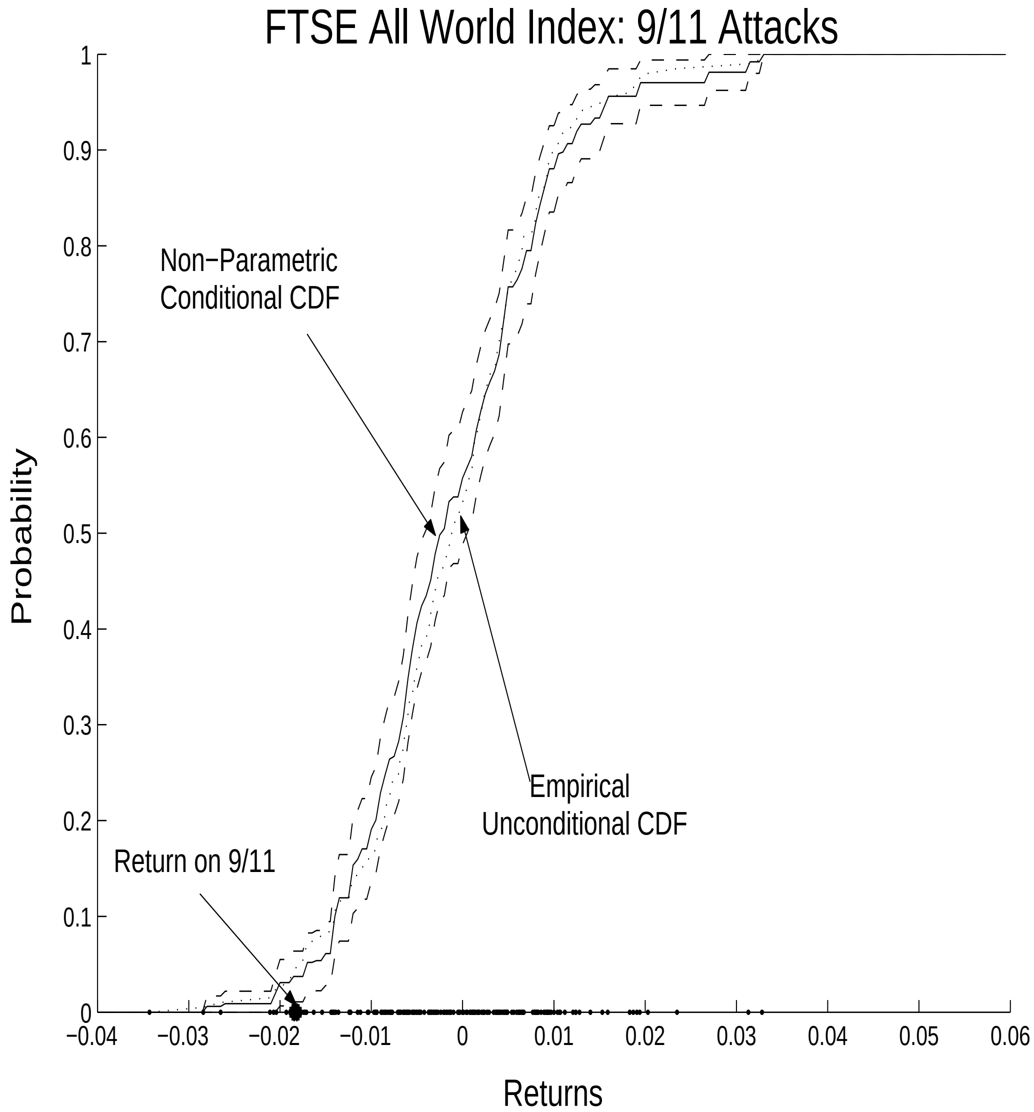

Figure 1. FTSE All World index: 9/11 attacks. A non-parametric conditional cumulative distribution function of returns on FTSE All World that we obtain based on 200 observations when conditioning is done on the return on 10th of September 2001, a day before the 9/11 attacks. The dotted line corresponds to an empirical unconditional cdf, dashed lines correspond to 95% confidence intervals.

Table 1. Impact of terrorist attacks, financial crashes and natural disasters on financial markets

	Terrorist Attacks		Financial Crashes		Natural Disasters	
Impact	Negative	Positive	Negative	Positive	Negative	Positive
FTSE All World	$\vee$		$\vee$		$\vee$	
MSCI Europe	$\vee$		$\vee$		$\vee$	
S&P 500	$\vee$		$\vee$		$\vee$	
SMI	$\vee$		$\vee$		$\vee$	
FTSE Global Banks	$\vee$		$\vee$		$\vee$	
FTSE Global Financials	$\vee$		$\vee$		$\vee$	
MSCI Europe Insurance	$\vee$		$\vee$		$\vee$	
FTSE All World Life Insurance	$\vee$		$\vee$		$\vee$	
FTSE All World Non-Life Insurance	$\vee$		$\vee$		$\vee$	
FTSE All World Travel	$\vee$		$\vee$		$\vee$	
MSCI Europe Airlines	$\vee$		$\vee$		$\vee$	
FTSE All World Aero/Defense	$\vee$	$\vee$	$\vee$		$\vee$	$\vee$
FTSE All World Pharma/Biotech	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$
FTSE All World Oil/Gas	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$
FTSE Europe Oil/Gas	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$
GSCI Commodity	$\vee$	$\vee$		$\vee$	$\vee$	$\vee$
GSCI Gold	$\vee$	$\vee$		$\vee$	$\vee$	$\vee$
J.P.Morgan GGBI	$\vee$	$\vee$		$\vee$	$\vee$	$\vee$
FTSE Eurozone Bond Index	$\vee$	$\vee$		$\vee$	$\vee$	$\vee$
FTSE US Government Bond Index	$\vee$	$\vee$			$\vee$	

Table 2. An event-study approach: Global and local effects. The table displays the effect of five major terrorist events on stock markets. It shows the strength of the impact of these extreme events. *AR* stands for the abnormal return and *CAR* for cumulative abnormal return. The number of stars next to AR and CAR indicates their statistical significance level: one star corresponds to 0.10, two stars to 0.05 and three stars to 0.01.

N	Event Day	Terrorist Attack	FTSE All World		MSCI Europe		S&P 500		SMI	
			Event-Day AR	6-day CAR	Event-Day AR	6-day CAR	Event-Day AR	6-day CAR	Event-Day AR	6-day CAR
1	11.09.2001	9/11 Attacks in	-0.0238^{***}	-0.0441^{**}	-0.0628^{***}	-0.0540^{**}	-0.0484^{***}	-0.0668^{***}	-0.0729^{***}	-0.0536^{**}
		the US	(-3.06)	(-2.23)	(-6.84)	(-2.32)	(-5.09)	(-2.77)	(-6.65)	(-1.92)
2	11.03.2004	Bombing in Madrid	-0.0174^{***}	-0.0185^{*}	-0.0262^{***}	-0.0491^{***}	-0.0160^{***}	-0.0057	-0.0295^{***}	-0.0500^{***}
			(-3.96)	(-1.67)	(-4.46)	(-3.68)	(-2.77)	(-0.40)	(-4.41)	(-3.20)
3	09.05.2004	Bombing in Russia	-0.0229^{***}	-0.0316^{***}	-0.0290^{***}	-0.0384^{**}	-0.0120^{*}	-0.0134	-0.0344^{***}	-0.0467^{***}
			(-4.64)	(-2.60)	(-4.55)	(-2.49)	(-1.60)	(-0.73)	(-5.30)	(-2.94)
4	07.10.2004	Bombing in Egypt	-0.0069^{*}	-0.0309^{***}	-0.0032	-0.0255^{**}	-0.0104^{**}	-0.0360^{**}	-0.0124^{**}	-0.0388^{***}
			(-1.62)	(-2.85)	(-0.64)	(-2.06)	(-1.74)	(-2.44)	(-2.18)	(-2.59)
5	07.07.2005	Suicide Bombing	-0.0050^{*}	0.0140	-0.0163^{***}	0.0044	0.0014	0.0252	-0.0108^{**}	0.0175
		in the UK	(-1.65)	(1.65)	(-3.58)	(0.36)	(0.36)	(2.22)	(-2.22)	(1.43)

Table 3. A non-parametric approach: Local and global effects. The table displays the effect of five major terrorist events on stock markets. CP stands for conditional probability and CAR stands for cumulative abnormal return. Abnormal movements in the event-day returns or in the post-event window correspond to the conditional probability in the interval (0.05;0.10]. Extreme index movements correspond to the conditional probability in the interval [0.00;0.05]. Standard errors are shown in parentheses.

N	Event Day	Terrorist Attack	FTSE All World				MSCI Europe			
			Event-day Return	CP	6-day CAR	CP	Event-day Return	CP	6-day CAR	CP
1	11.09.2001	9/11 Attacks in	-0.0183^{**}	0.035	-0.0441^{**}	0.04	-0.0635^{***}	0.00	-0.0540^{**}	0.011
		the US		(0.013)		(0.016)		(0.000)		(0.019)
2	11.03.2004	Bombing in Madrid	-0.0170^{***}	0.00	-0.0185	0.29	-0.0261^{***}	0.00	-0.0491^{**}	0.03
				(0.000)		(0.032)		(0.000)		(0.012)
3	09.05.2004	Bombing in Russia	-0.0212^{***}	0.00	-0.0316	0.11	-0.0265^{***}	0.00	-0.0384^{*}	0.08
				(0.000)		(0.022)		(0.000)		(0.020)
4	07.10.2004	Bombing in Egypt	-0.0060^{*}	0.09	-0.0309^{*}	0.10	-0.0014	0.41	-0.0255	0.24
				(0.020)		(0.024)		(0.036)		(0.031)
5	07.07.2005	Suicide Bombing	-0.0034	0.21	0.0140	0.82	-0.0146^{**}	0.02	0.0044	0.62
		in the UK		(0.033)		(0.027)		(0.009)		(0.036)

N	Event Day	Terrorist Attack	S&P 500				SMI			
			Event-day Return	CP	6-day CAR	CP	Event-day Return	CP	6-day CAR	CP
1	11.09.2001	9/11 Attacks in	-0.0505^{***}	0.00	-0.0668^{**}	0.03	-0.0733^{***}	0.00	-0.0536	0.14
		the US		(0.000)		(0.013)		(0.000)		(0.026)
2	11.03.2004	Bombing in Madrid	-0.0153^{***}	0.003	-0.0057	0.41	-0.0294^{***}	0.00	-0.0500^{**}	0.03
				(0.004)		(0.036)		(0.000)		(0.012)
3	09.05.2004	Bombing in Russia	-0.0106^{*}	0.07	-0.0134	0.33	-0.0324^{***}	0.00	-0.0467^{**}	0.04
				(0.019)		(0.035)		(0.000)		(0.013)
4	07.10.2004	Bombing in Egypt	-0.0100^{*}	0.06	-0.0360^{*}	0.08	-0.0106	0.14	-0.0388^{*}	0.09
				(0.018)		(0.023)		(0.026)		(0.020)
5	07.07.2005	Suicide Bombing	0.0024	0.68	0.0252	0.89	-0.0093^{**}	0.03	0.0175	0.82
		in the UK		(0.033)		(0.022)		(0.012)		(0.027)

Table 4. Comparison of the event-day negative impact of terrorist attacks, financial crashes and natural catastrophes across different methodologies. The table displays the event-day negative effect of terrorist attacks, financial crises and natural catastrophes on stock markets. For example, for FTSE All World index the results of an event-study with respect to terrorist attacks are as follows: out of 77 events under consideration, 30 had an event-day negative impact. For 15 events out of 30, index return movements were extreme.

Index	Terrorist Attacks									Financial Crashes								
	Event-Study			Non-Parametric			GARCH-EVT			Event-Study			Non-Parametric			GARCH-EVT		
	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events
FTSE All World	30	15	77	28	19	77	26	19	76	2	2	4	2	2	4	1	1	4
MSCI Europe	29	22	77	28	17	77	22	16	77	1	1	4	1	1	4	1	1	4
S&P 500	19	14	77	23	19	77	22	15	77	2	2	4	2	2	4	2	2	4
SMI	27	23	77	29	22	77	28	15	77	1	1	4	1	1	4	1	1	4
FTSE Global Banks	20	13	67	21	11	66	20	10	69	2	1	3	1	1	3	0	0	2
FTSE Global Financials	23	17	67	23	19	66	21	19	66	2	2	3	1	1	3	2	1	3
MSCI Europe Insurance	29	20	75	31	18	73	28	20	73	1	1	3	1	1	3	1	1	3
FTSE All World Life Ins.	26	20	77	27	17	76	23	11	74	2	2	4	1	1	4	2	2	4
FTSE All World Non-Life Insurance	23	13	77	24	19	76	22	15	75	2	2	4	1	1	4	1	1	3
FTSE All World Travel	19	12	77	20	15	76	19	8	77	2	2	4	2	2	4	2	2	4
MSCI Europe Airlines	25	20	75	21	19	73	23	21	74	1	1	3	1	1	3	1	1	3
FTSE All World Aero/Def.	21	16	77	21	15	76	23	16	76	2	2	4	2	2	4	2	2	4
FTSE All World Pharma/Bio	20	15	77	18	15	76	15	10	76	2	2	4	2	2	4	2	2	4
FTSE All World Oil/Gas	13	10	77	16	12	76	13	12	62	1	1	4	1	1	4	0	0	2
FTSE Europe Oil/Gas	10	10	77	13	8	76	10	8	62	1	1	4	1	0	4	0	0	2
GSCI Commodity	8	5	77	10	7	77	7	2	77	0	0	4	0	0	4	0	0	4
GSCI Gold	9	7	77	8	6	77	9	7	77	0	0	4	0	0	4	0	0	4
J.P.Morgan GGBI	4	3	77	4	4	77	4	3	77	0	0	4	0	0	4	0	0	4
FTSE Eurozone Bond Index	4	3	63	4	3	62	3	2	60	1	1	2	0	0	2	0	0	1
FTSE US Gov. Bond Index	7	2	59	6	1	59	3	1	59	0	0	1	0	0	1	0	0	1

Table 4: (cont'd.)

Index	Natural Catastrophes Event-Study			Natural Catastrophes Non-Parametric			Natural Catastrophes GARCH-EVT		
	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events
FTSE All World	1	0	18	2	1	18	0	0	18
MSCI Europe	3	2	19	2	2	19	2	1	19
S&P 500	0	0	19	0	0	19	0	0	19
SMI	1	1	19	1	1	19	1	1	19
FTSE Global Banks	1	0	17	0	0	16	0	0	17
FTSE Global Financials	0	0	17	1	1	16	0	0	17
MSCI Europe Insurance	2	2	17	2	2	17	2	0	17
FTSE All World Life Insurance	1	0	18	1	0	18	0	0	17
FTSE All World Non-Life	3	1	18	3	0	18	2	0	16
FTSE All World Travel	2	1	18	0	0	18	0	0	18
MSCI Europe Airlines	1	1	17	0	2	17	2	1	17
FTSE All World Aero/Defense	0	0	18	1	1	18	1	0	18
FTSE All World Pharma/Biotech	0	0	18	0	0	18	0	0	18
FTSE All World Oil/Gas	1	0	18	0	0	18	0	0	15
FTSE Europe Oil/Gas	2	1	18	1	1	18	1	0	15
GSCI Commodity	4	4	19	3	2	19	2	2	19
GSCI Gold	0	0	19	1	0	19	0	0	19
J.P.Morgan GGBI	3	3	19	2	1	19	2	2	19
FTSE Eurozone Bond Index	2	0	15	2	1	14	1	0	14
FTSE US Government Bond Index	0	0	9	1	0	8	0	0	9

Table 5. Comparison of the event-day positive impact of terrorist attacks, financial crashes and natural catastrophes across different methodologies. The table displays the event-day positive effect of terrorist attacks, financial crises and natural catastrophes on stock markets. For example, for FTSE All World Aero/Defence index the results of an event-study with respect to terrorist attacks are as follows: out of 77 events under consideration, 6 had an event-day positive impact. For 4 out of 6 events, index return movements were extreme.

Index	Terrorist Attacks									Financial Crashes								
	Event-Study			Non-Parametric			GARCH-EVT			Event-Study			Non-Parametric			GARCH-EVT		
	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events
FTSE All World Aero/Def.	6	4	77	7	5	76	6	6	76	0	0	4	0	0	4	0	0	4
FTSE All World Pharma/ Biotech	6	1	77	6	5	76	9	6	76	0	0	4	1	1	4	1	1	4
FTSE All World Oil/Gas	1	0	77	7	1	76	4	1	62	0	0	4	0	0	4	0	0	2
FTSE Europe Oil/Gas	2	0	77	5	2	76	6	3	62	0	0	4	1	1	4	0	0	2
GSCI Commodity	7	4	77	8	7	77	10	5	77	0	0	4	0	0	4	0	0	4
GSCI Gold	4	2	77	6	3	77	2	2	77	2	1	4	1	1	4	2	1	4
J.P.Morgan GGBI	13	9	77	10	8	77	12	5	77	2	2	4	2	2	4	2	2	4
FTSE Eurozone BI	14	7	63	13	7	62	12	8	60	0	0	2	0	0	2	0	0	1
FTSE US Gov. Bond Index	8	3	59	13	5	59	7	4	59	0	0	1	0	0	1	0	0	1

Table 5: (cont'd.)

	Natural Catastrophes								
Index	Event-Study			Non-Parametric			GARCH-EVT		
	Total	Ext.	Num. Events	Total	Ext.	Num. Events	Total	Ext.	Num. Events
FTSE All World Aero/Defense	2	2	18	1	1	18	1	1	18
FTSE All World Pharma/Biotech	0	0	18	0	0	18	0	0	18
FTSE All World Oil/Gas	0	0	18	0	0	18	1	0	15
FTSE Europe Oil/Gas	0	0	18	1	0	18	2	0	15
GSCI Commodity	3	2	19	3	2	19	2	2	19
GSCI Gold	1	1	19	2	2	19	0	0	19
J.P.Morgan GGBI	3	3	19	2	1	19	1	0	19
FTSE Eurozone Bond Index	2	1	15	2	1	14	2	2	14
FTSE US Government Bond Index	0	0	9	0	0	8	0	0	9

Table 6. Positive effect on the aero/defense and pharma/biotech industries: Common terrorist events.

			FTSE All World Aero/Defense			FTSE All World Pharma/Biotech		
N	Date	Attack	Event Study	Non-Parametric A.	GARCH-EVT	Event Study	Non-Parametric A.	GARCH-EVT
1	19.04.1995	Bombing in Oklahoma City	∨	∨	∨			
2	08.01.1996	Kidnapping in Indonesia	∨	∨	∨			
3	31.01.1996	Bomb Attack in Sri Lanka				∨	∨	
4	04.03.1996	Suicide Bombing in Israel	∨	∨	∨	∨	∨	∨
5	10.04.2002	Armed Assault in India	∨	∨	∨		∨	∨
6	10.04.2002	Suicide Bombing in Israel	∨	∨	∨		∨	∨
7	08.05.2002	Bombing in Pakistan	∨	∨	∨		∨	∨
8	17.07.2002	Suicide Bombing in Israel	∨	∨				
9	04.08.2002	Bombing in Israel				∨	∨	
10	05.08.2002	Armed Assault in Pakistan				∨	∨	
11	12.10.2002	Bombing in Indonesia/Bali				∨	∨	∨
12	27.12.2002	Suicide Bombing in Russia				∨	∨	
13	01.02.2004	Suicide Bombing in Iraq				∨	∨	∨
14	15.04.2005	Armed Assault in Colombia				∨	∨	∨

Table 7. Positive effect on the oil/gas industry: Common terrorist events.

			FTSE Europe Oil/Gas			FTSE World Oil/Gas		
N	Date	Attack	Event Study	Non-Parametric A.	GARCH-EVT	Event Study	Non-Parametric A.	GARCH-EVT
1	17.04.1999	Bombing in the UK		$\vee$	$\vee$		$\vee$	
2	11.09.2001	9/11 Attacks in the US	$\vee$	$\vee$	$\vee$	$\vee$		
3	08.05.2002	Bombing in Pakistan	$\vee$			$\vee$	$\vee$	$\vee$
4	17.07.2002	Suicide Bombing in Israel	$\vee$	$\vee$		$\vee$	$\vee$	
5	31.07.2002	Bombing in Israel		$\vee$		$\vee$	$\vee$	
6	04.08.2002	Bombing in Israel		$\vee$		$\vee$	$\vee$	
7	05.08.2002	Armed Assault in Pakistan		$\vee$		$\vee$	$\vee$	
8	05.08.2003	Bombing in Indonesia/Bali	$\vee$	$\vee$		$\vee$	$\vee$	
9	05.12.2003	Suicide Bombing in Russia	$\vee$			$\vee$		
10	09.12.2003	Suicide Bombing in Russia	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$
11	18.04.2005	Armed Assault in Russia				$\vee$	$\vee$	
12	25.08.2005	Bombing in Russia		$\vee$		$\vee$	$\vee$	

Table 8. Portfolio performance analysis of the Swiss investors. The table reports the portfolio performance measured via Sharpe Ratio of the SMI Index and two different hedged SMI portfolios. The portfolios were constructed 1 week before the September 11 terrorist attacks and performance is evaluated on the given dates.

	11.09.2001	18.09.2001	25.09.2001	4.10.2001
First Investor	-0.7161	-0.3448	-0.2360	-0.0942
Second Investor	-0.5264	0.0396	0.0002	0.1830
Third Investor	-0.0125	0.5982	0.3493	0.3753

Table 9. Portfolio performance analysis of the European investors. The table reports the portfolio performance measured via Sharpe Ratio of the MSCI Europe Index and two different hedged MSCI Europe Index portfolios. The portfolios were constructed 1 week before the September 11 terrorist attacks and performance is evaluated on the given dates.

	11.09.2001	18.09.2001	25.09.2001	4.10.2001
First Investor	-0.8497	-0.4266	-0.3308	-0.1234
Second Investor	-0.5506	0.0395	-0.0097	0.1852
Third Investor	-0.3199	0.4796	0.2749	0.3573

Table 10. Robustness checks for the non-parametric approach: SMI index returns. The table displays the effect of ten terrorist events on robustified and unrobustified SMI index returns. CP stands for conditional probability and CAR stands for cumulative abnormal return. Abnormal movements in event-day returns or in the post-event window correspond to the conditional probability in the interval (0.05;0.10]. Extreme index movements correspond to the conditional probability in the interval [0.00;0.05]. Standard errors are shown in parentheses. NA stands for "Not Applicable" i.e. couldn't be computed due to data restrictions.

N	Event Day	Terrorist Attack	Robustified SMI				Unrobustified SMI			
			Return	CP on Aver.	6-day CAR	CP	Return	CP on Aver.	6-day CAR	CP
1	11.09.2001	9/11 Attacks in the US	-0.0379^{***}	0.00 (0.000)	NA	NA	-0.0733^{***}	0.00 (0.000)	-0.0536	0.14 (0.026)
2	11.03.2004	Bombing in Madrid	-0.0154^{***}	0.00 (0.000)	NA	NA	-0.0294^{***}	0.00 (0.000)	-0.0500^{**}	0.03 (0.012)
3	09.05.2004	Bombing in Russia	-0.0184^{***}	0.00 (0.000)	NA	NA	-0.0324^{***}	0.00 (0.000)	-0.0467^{**}	0.04 (0.013)
4	07.10.2004	Bombing in Egypt	-0.0073	0.1457 (0.0259)	NA	NA	-0.0106	0.14 (0.026)	-0.0388^{*}	0.09 (0.020)
5	18.04.2005	Armed Assault in Russia	-0.0029	0.2718 (0.0316)	-0.0047	0.3278 (0.0386)	-0.0157^{**}	0.02 (0.009)	-0.0196	0.18 (0.027)
6	10.05.2005	Bombing in Russia	-0.0066	0.1584 (0.0259)	-0.0206	0.1254 (0.0234)	-0.0143^{**}	0.04 (0.014)	-0.007	0.38 (0.04)
7	24.06.2005	Armed Assault in Israel	-0.0032	0.2767 (0.0321)	-0.0167	0.1439 (0.0253)	-0.0081	0.1 (0.022)	-0.0133	0.3 (0.033)
8	07.07.2005	Suicide Bombing in the UK	-0.0033	0.2623 (0.0322)	0.0268	0.8992 (0.0200)	-0.0093^{**}	0.03 (0.012)	0.0175	0.82 (0.027)
9	23.07.2005	Bombing in Egypt	-0.0032	0.3517 (0.0444)	0.0186	0.8369 (0.0271)	0.0018	0.57 (0.035)	0.0119	0.73 (0.032)
10	15.08.2005	Bombing in Russia and Egypt	-0.0053	0.1794 (0.0284)	-0.0288^{**}	0.0400 (0.0136)	-0.0017	0.41 (0.039)	-0.0195	0.21 (0.03)

Table 11. Robustness checks for the non-parametric approach: MSCI Europe index returns. The table displays the effect of ten terrorist events on robustified and unrobustified MSCI Europe index returns. CP stands for conditional probability and CAR stands for cumulative abnormal return. Abnormal movements in event-day returns or in the post-event window correspond to the conditional probability in the interval (0.05;0.10]. Extreme index movements correspond to the conditional probability in the interval [0.00;0.05]. Standard errors are shown in parentheses. NA stands for "Not Applicable" i.e. couldn't be computed due to data restrictions.

N	Event Day	Terrorist Attack	Robustified MSCI Europe				Unrobustified MSCI Europe			
			Return	CP on Aver.	6-day CAR	CP	Return	CP on Aver.	6-day CAR	CP
1	11.09.2001	9/11 Attacks in the US	-0.0253^{***}	0.00 (0.000)	NA	NA	-0.0635^{***}	0.00 (0.000)	-0.0540^{**}	0.011 (0.019)
2	11.03.2004	Bombing in Madrid	-0.0074^{*}	0.059 (0.0193)	NA	NA	-0.0261^{***}	0.00 (0.000)	-0.0491^{**}	0.03 (0.012)
3	09.05.2004	Bombing in Russia	-0.0114^{**}	0.0312 (0.0124)	NA	NA	-0.0265^{***}	0.00 (0.000)	-0.0384^{*}	0.08 (0.020)
4	07.10.2004	Bombing in Egypt	0.0019	0.6487 (0.0375)	NA	NA	−0.0014	0.41 (0.036)	−0.0255	0.24 (0.031)
5	18.04.2005	Armed Assault in Russia	−0.0029	0.3015 (0.0353)	0.0097	0.7269 (0.0344)	-0.0178^{***}	0.04 (0.013)	−0.0070	0.50 (0.036)
6	10.05.2005	Bombing in Russia	0.0001	0.4097 (0.0356)	−0.0155	0.1410 (0.0235)	−0.0060	0.12 (0.024)	0.0043	0.54 (0.040)
7	24.06.2005	Armed Assault in Israel	−0.0019	0.4560 (0.0390)	−0.0035	0.4466 (0.0396)	-0.0075^{*}	0.06 (0.017)	−0.0055	0.46 (0.035)
8	07.07.2005	Suicide Bombing in the UK	-0.0088^{**}	0.0377 (0.0141)	0.0165	0.9069 (0.0204)	-0.0146^{**}	0.02 (0.009)	0.0044	0.62 (0.036)
9	23.07.2005	Bombing in Egypt	−0.0040	0.2409 (0.0304)	0.0051	0.6296 (0.0347)	0.0026	0.59 (0.035)	0.0008	0.55 (0.036)
10	15.08.2005	Bombing in Russia and Egypt	−0.0063	0.1206 (0.0294)	-0.0221^{*}	0.0930 (0.0233)	−0.0010	0.31 (0.033)	−0.0057	0.44 (0.035)

References

Abadie, A., Gardeazabal, J., 2003. The economic cost of conflict: A case study of the basque country. American Economic Review 93, 113–132.

Arin, K., Ciferri, D., Spagnolo, N., 2008. The price of terror: The effects of terrorism on stock market returns and volatility. Economics Letters 101, 164–167.

Bali, T., Mo, H., Tang, Y., 2008. The role of autoregressive conditional skewness and kurtosis in the estimation of conditional var. Journal of Banking and Finance 32, 269–282.

Bonturi, M., Koen, V., Lenain, P., 2002. Economic consequences of terrorism. Working Paper. OECD Economics Department.

Brown, S., Warner, J., 1980. Measuring security price performance. Journal of Financial Economics , 205–258.

Carter, D., Simkins, B., 2001. Do markets react rationally? The effect of the September 11 tragedy on airline stock returns. Working Paper. Oklahoma State University.

Chen, A., Siembs, T., 2004. The effects of terrorism on global capital markets. European Journal of Political Economy 20, 349–366.

Christoffersen, P., Hahn, J., Inoue, A., 2001. Testing and comparing value at risk measures. Journal of Empirical Finance 8, 325–342.

DeMiguel, V., Garlappi, L., Uppal, R., 2009. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? Review of Financial Studies 22, 1915–1953.

Eldor, R., Melnick, R., 2004. Financial markets and terrorism. European Journal of Political Economy 20, 367–386.

Elton, E., Gruber, M., Urich, T., 1978. Are betas best? Journal of Finance , 1375–1384.

Enders, W., Sandler, T., Parise, G., 1992. An econometric analysis of the impact of terrorism on tourism. Kyklos 45, 145–165.

Fan, J., Yao, Q., 2003. Nonlinear Time Series. Nonparametric and Parametric Methods. Springer: New York.

Farinelli, S., Ferreira, M., Rossello, D., Thoeny, M., Tibiletti, L., 2008. Beyond sharpe ratio: Optimal asset allocation using different performance ratios. Journal of Banking & Finance 32, 2057–2063.

FCO, 2005. Risk of terrorism: Significant terrorist attacks. http://www.fco.gov.uk/. Foreign and Commonwealth Office, UK.

Jha, R., Korkie, B., Turtle, H., 2009. Measuring performance in a dynamic world: Conditional mean-variance fundamentals. Journal of Banking & Finance 33, 1851–1859.

Johnston, R., Nedelescu, O., 2005. The impact of terrorism on financial markets. Working Paper. Washington: International Monetary Fund.

Karolyi, G., 2006. Shock markets: What do we know about terrorism and the financial markets? Canadian Investment Review .

Karolyi, G., Martell, R., 2006. Terrorism and the stock market. Working Paper. The Ohio State University.

Kuester, K., Mittnik, S., Paolella, M., 2006. Value-at-risk prediction: A comparison of alternative strategies. Journal of Financial Econometrics 4, 53–89.

Longin, F., 2005. The choice of the distribution of asset returns: How extreme value theory can help? Journal of Banking and Finance 29, 1017–1035.

McNeil, A., Frey, R., 2000. Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach. Journal of Empirical Finance 7, 271–300.

Raby, G., 2003. The costs of terrorism and the benefits of cooperating to combat terrorism. Paper. Department of Foreign Affairs and Trade, Economic Analytical Unit.

TRA, 2000. Significant events. http://www.terrorism.com/. Terrorism Research Center.

Wang, J., Meric, G., Liu, Z., Meric, I., 2009. Stock market crashes, firm characteristics, and stock returns. Journal of Banking and Finance 33, 1563–1574.

Zakamouline, V., Koekebakker, S., 2009. Portfolio performance evaluation with generalised sharpe ratios: Beyond the mean and variance. Journal of Banking and Finance 33, 1242–1254.

Part III

Variance Swaps, Risk Premiums, and Expectation Hypothesis

Yacine Aït-Sahalia, Mustafa Karaman, and Loriano Mancini

Abstract

We study the term structure of actual variance swap (VS) rates, which are popular volatility derivative contracts. When time to maturity increases, level and persistence of VS rates increase, while volatility and higher order moments decrease. A model-free approach detects a significant jump risk component in VS rates. The term structure of variance risk premium is negative and generally downward sloping, while the component due to negative jumps exhibit similar features in quite times but is upward sloping in turbulent times. Theoretically, Expectation Hypothesis does not hold but bias and inefficiency are modest for short time to maturities. Simple trading strategies with VS's yield significant returns.

1. Introduction

During the last decade the interest of investors in volatility derivative products has grown enormously. Perhaps the most manifest example of this phenomenon is the attention devoted to the VIX volatility index of CBOE, sometimes called "market's fear gauge", see Whaley (2000). In March 2004 the CBOE introduced futures and in February 2006 European options on VIX. Nowadays these contracts are among the most actively traded contracts at CFE and CBOE. Such demand is certainly driven by the large impact of stochastic variance on asset returns, and thus portfolio and risk management.

Among all volatility derivative products, the variance swap (VS) contract is the most direct way to achieve exposure to or hedge against variance risk. The long position in VS receives the realized variance over a certain time horizon and pays a fixed rate, called VS rate. Thus the payoff is positive when variance increases over the given time horizon. In contrast to options, variance swap contracts do not carry additional risk exposures, such as delta exposure. According to some financial press, e.g., Gangahar (2006), VS has become the preferred tool by hedge funds and proprietary traders to bet on volatility directions. VS's are traded over the counter on various underlying assets (such as equity indices, exchange/interest rates, or commodities), and at many different maturities (sometimes as long as 10 years). Some estimates reported in 2006 showed that the daily trading volume in equity index VS reached USD 4–5 million vega notional; Jung (2006). This notional corresponds to payments of more than USD 1 billion per volatility percentage unit on an annual base.

Our goal is to study the risk premiums and information content embedded in term structure of VS contracts on the S&P 500 index. Such variance swaps are among the most important contracts in the arena. We use actual (not synthetic) daily VS rates on the S&P 500 index with fixed time to maturity of 2-, 3-, 6-, 12- and 24-month from January 4, 1996 to September 2, 2010, and quoted by a major broker dealer in New York City.

Data-based analysis of VS rates reveals the following phenomena. When time to maturity increases (i) level and persistence of VS rates increase while volatility, skewness and kurtosis decrease. (ii) Large jump risk component is embedded in VS and not only during market crashes. We use a model-free method to measure jump component in VS rates, relying on recent theoretical results in the so-called model-free implied volatility literature. Specifically, we compare variance swap rates and VIX-type indices calculated using SPX options for various maturities. (iii) Principal Component Analysis shows that two factors, which can be interpreted as level and slope factors, explain 99.8% of variation in VS rates.

Various aspects of the term structure of VS rates cannot be studied in a model-free way because necessary data are either insufficient or simply unavailable. For example model-free analysis of the term structure of priced jump risk in VS would require long lived, out-of-the-money, SPX options with a fixed time to maturity. These options are generally unavailable. Available options present periodic, saw-toothed patterns time to maturities. Usual interpolation schemes of discrete time to maturities are likely to introduce errors. Hence to deepen the analysis of VS term structure, we introduce a parametric model for VS rates. Empirical features (i) and (iii) above suggest that two-factor stochastic volatility model is necessary. Empirical feature (ii) suggests to include a jump component, with stochastic intensity, in the stock price dynamic. The corresponding parametric model is a two-factor stochastic volatility with stochastic jump intensity model, which is adopted in our subsequent analysis. Although many estimation methods are available, accurate estimation is challenging. However, a key feature of stochastic volatility models with affine drift is that model-based VS rates are affine in latent state variables. This feature suggests to filter out latent states directly from VS rates. Then we use a likelihood-based method to estimate the model. The state vector follows a multivariate jump-diffusion stochastic volatility process and thus its transition density is unknown. Since jumps in stock prices are important but rare events, using Bayes rule we approximate the transition density of the state vector with a mixture of no-jump and 1-jump densities. Such densities are unknown as well but can be accurately approximated using the closed-form likelihood expansion method. The latter was introduced and developed in the univariate setting by Aït-Sahalia (2002a), Aït-Sahalia (1999), and extended to the multivariate setting by Aït-Sahalia (2008), Aït-Sahalia and Kimmel (2007), Aït-Sahalia and Kimmel (2010). Monte Carlo simulation shows that likelihood-based estimation is accurate at daily frequency.

Real data estimation based on time series S&P 500 index return data and cross sectional variance swap data reveals that the two-factor stochastic volatility with stochastic jump intensity model fits the data well. In particular, in- and out-of-sample pricing errors of VS are small. The model allows to uncover the following features of VS rates. Term structures of (i) variance risk premium (VRP) is negative and usually downward sloping; (ii) variance risk premium due to negative jumps is negative, generally downward sloping in quite times but upward sloping during market crashes; (iii) variance risk premium due to positive jumps tends to mirror the one of negative jumps. (iv) Theoretical analysis shows that Expectation Hypothesis does not hold because of the various risk premiums. However actual biases and inefficiencies are modest for short/medium time to maturities, say below one year, suggesting that VS rates are good predictors of future realized variance. Finding (i) implies that the longer the time to maturity the higher the variance risk premium. Ex-ante, always negative, VRP appears to be different from ex-post, sometimes positive, VRP based on realized variances. This finding complements model-free, single maturity results in Carr and Wu (2009) as we study model-based, term structure of variance risk premium. Findings (ii) and (iii) suggest that the contribution of jump component to variance risk premium is modest in quite times, but it becomes substantial during market crashes, and mostly impacts the short-end of the term structure of VS rates. These findings complement model-free, single maturity results in Bollerslev and Todorov (2011) as we study model-based, term structure of variance risk premium due to jump component.

Finally, we use the estimation results for a simple but model-consistent investment strategy on VS's. Everyday, we short a VS if the expected profit from this strategy is larger than a threshold level. On those days we also invest \$1 to S&P 500 index and liquidate the position when VS matures for comparison. The results show that, VS investments using the estimation results outperform long S&P 500 investments for most of the time.

2. Variance Swaps

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying usual conditions (e.g., Protter (2005)), with P denoting the objective probability measure. Let S be a semimartingale modeling the stock price or index process with dynamic

$$dS_t/S_{t_-} = \mu_t\, dt + \sqrt{v_t}\, d\tilde{W}_t^P + (\exp(J_t^P) - 1)\, dN_t^P - \nu_t^P\, dt \tag{1}$$

where μ_t is the drift, v_t the spot variance, $d\tilde{W}_t^P$ the Brownian increment, dN_t^P the jump process with stochastic intensity λ_t and jump size one, J_t^P the random jump size, and $\nu_t^P = g^P \lambda_t$ the compensator with $g^P = E[\exp(J_t^P) - 1]$. When a jump occurs the induced price change is $(S_t - S_{t_-})/S_{t_-} = \exp(J_t^P) - 1$ which implies that $\log(S_t/S_{t_-}) = J_t^P$, hence J_t^P is the jump size of the log-price. At this stage, the dynamics of drift, variance, and jump component are left unspecified because the first part of the analysis of VS contracts will be model-free. Thus Model (1) virtually subsumes all models commonly used in Finance. Later on we will put a parametric structure on the model to deepen the analysis of VS contracts.

Let $t = t_0 < t_1 < \cdots < t_n = t+\tau$ denote the trading days for a given time period $[t, t+\tau]$, for e.g., six months. The annualized realized variance is the annualized sum of squared log-returns over the time horizon $[t, t+\tau]$

$$\mathrm{RV}_{t,t+\tau} = \frac{252}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2. \tag{2}$$

A long position in a variance swap contract payoffs at time $t+\tau$ the difference between the realized variance, $\mathrm{RV}_{t,t+\tau}$, and the variance swap rate, $\mathrm{VS}_{t,t+\tau}$, fixed at time t, times a notional amount used to convert the payoff in dollar terms:

$$(\mathrm{RV}_{t,t+\tau} - \mathrm{VS}_{t,t+\tau}) \times (\text{notional amount}).$$

The analysis of variance swap contracts is simplified when the realized variance is replaced by the quadratic variation of the log-price process. It is known that when $\sup_{i=1,\ldots,n}(t_i - t_{i-1}) \to 0$ the realized variance in Equation (2) converges in probability to the quadratic variation of the log-price, $\mathrm{QV}_{t,t+\tau}$, (e.g., Jacod and Protter (1998)):

$$\frac{252}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \longrightarrow \frac{1}{\tau} \int_t^{t+\tau} v_s \, ds + \frac{1}{\tau} \sum_{u=N_t}^{N_{t+\tau}} J_u^2 = \mathrm{QV}_{t,t+\tau}.$$

This approximation is commonly adopted in practice (e.g., Egloff, Leippold, and Wu (2010)) and justified for daily sampling frequency (e.g., Broadie and Jain (2008) and Jarrow, Kchia, Larsson, and Protter (2011)) as is the case in our dataset. If the spot variance has a jump component the convergence above still holds and such jumps in variance are accommodated in the time integral of v_s.

As usual we assume absence of arbitrage which implies the existence of an equivalent risk-neutral measure Q. By convention the variance swap contract has zero value at inception. Assuming that the interest rate does not depend on the quadratic variation, which is certainly a tenuous assumption, no arbitrage implies

$$\mathrm{VS}_{t,t+\tau} = E_t^Q[\mathrm{QV}_{t,t+\tau}] \tag{3}$$

where E_t^Q denotes time-t conditional expectation under Q. The variance swap rate depends on information or market conditions at time t, as well as on time to maturity τ, which produces the term structure we are interested in.

2.1. *Dataset*

Our dataset consists of over the counter quotes on variance swap rates on the S&P 500 index provided by a major broker-dealer in New York City. The data are daily closing quotes on variance swap rates with fixed time to maturities at 2, 3, 6, 12, and 24 months from January 4, 1996 to September 2, 2010, that are 3,624 observations for each maturity.[1] Figure 1 shows the term structure of VS rates over time and suggests that VS rates are mean-reverting, volatile, with spikes and clustering during the major financial crisis over the last 15 years, and historically highest values during the recent Subprime crisis. While most term structures are upward sloping (53% of our sample), they are often ∪-shape too (23% of our sample). The remaining term structures are roughly split in downward sloping and ∩-shape term structures.[2]

The bottom and pick, respectively, of the ∪- and ∩-shape term structures can be anywhere at 3 or 6 or 12 months to maturity VS rate. The slope of the term structure (measured as the difference between 24 and 2 months VS rates) shows a strong negative association with the contemporaneous level volatility. Thus during high volatility periods or financial crisis, the short-end of the term structure (VS rates with 2 or 3 months to maturity) rises more than the long-end producing downward sloping term structures.

Table 1 provides summary statistics of our data. For the sake of interpretability we follow market practice and report variance swap rates in volatility percentage units, i.e. $\sqrt{\mathrm{VS}_{t,t+\tau}}\times 100$. Various features clearly emerge. The mean level and first order autocorrelation of swap rates are slightly but monotonically increasing with time to maturity. The standard deviation, skewness and kurtosis of swap rates are strictly decreasing with time to maturity. Ljung–Box tests strongly reject the hypothesis of zero autocorrelations, while generally Dickey–Fuller tests do not detect unit roots,[3] except for longest maturities—although it is well-known that the outcome of standard unit root tests should be carefully interpreted with slowly decaying memory processes; see, for instance, Schwert (1987). First order autocorrelations of swap rates range between 0.982 and 0.995, confirming mean reversion in these series. As these coefficients increase with time to maturity, the longer the maturity the higher the persistence of VS rates with mean half-life[4] of shocks between 38 and 138 days.

Principal Component Analysis (PCA) shows that the first principal component explains about 95.4% of the total variance of VS rates and can be interpreted as a level factor, while the second principal component explains an additional 4.4% and can be interpreted as a slope

[1]Various statistical analysis do not detect any week effect so we use all available daily data.

[2]On some occasions, the term structure is $\sim$-shape but the differences between the 2 and 3 months VS rates are virtually zero and those term structures are best seen as ∪-shape.

[3]Under the null hypothesis of unit root the Dickey–Fuller test statistic has zero expectation.

[4]The half-life H is defined as the time necessary to halve a unit shock and solves $\varrho^H = 0.5$ where ϱ is the first order autocorrelation coefficient.

factor. This finding is somehow expected because PCA of several other term structures, such as interest rate yields, produce qualitatively similar results. Less expected is that these two factors explain nearly all the variation in VS rates, precisely 99.8% of variance of VS rates. Repeating the PCA for various subsamples reveals little variation in the first two factors and explained total variance. Overall, PCA suggests that at most two factors are driving VS rates.

Table 1 also shows summary statistics of ex-post realized variance of S&P 500 index returns for various time to maturities. All statistics of realized variance share qualitatively the same features as those of VS rates. The main differences are that realized variances are substantially lower and more volatile, positively skewed and leptokurtic than VS rates. These differences highlight the large negative variance risk premiums as well as the profitability and riskiness of shorting VS contracts. Since realized variances over various horizons are lower than VS rates on average, the negative risk premium documented by Carr and Wu (2009) for 30-day (synthetic) variance swap rates also extends to long maturity VS rates. Although shorting VS contracts produces on average positive payoffs at maturity, the large variability and in particular the positive skewness of ex-post realized variances can induce large losses to the short side of the contract.

2.2. Model-free Jump Component in Variance Swap Rates

We use a model-free method to quantify the priced jump component in VS rates. We take advantage of recent theoretical advances collectively described as model-free implied volatility literature; see e.g., Carr and Madan (1998), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), and Carr and Wu (2009). The main result in this literature is that if the stock price process is continuous, then the variance swap payoff at a given maturity can be exactly replicated by dynamically trading in futures and a static position in a continuum of European options with different strikes and same maturity. The replication is model-free in the sense that the stock price can follow the general model (1) but with the restriction $\lambda_t = 0$. If indeed the stock price has a jump component, then this replication no longer holds. Thus comparing variance swap rates and the cost of replicating portfolio allows to gauge whether or not VS rates embed a priced jump component. In practice, of course, only a typically small number of options is available. Interpolation and extrapolation of strike prices are necessary to compute the cost of the replicating portfolio. Moreover, options are available only for a few maturities and it is unlikely that options expiring exactly on the required maturity are available. Hence an interpolation across maturities is required as well. Such interpolations and extrapolations introduce an approximation error in the cost of replicating portfolio; see Jiang and Tian (2005) for a detailed discussion of these issues.

It is known that the quadratic variation can be represented as follows

$$\mathrm{QV}_{t,t+\tau} = \frac{2}{\tau}\left[\frac{F_{t+\tau}}{F_t} - 1 - \log\frac{F_{t+\tau}}{F_t}\right] + \frac{2}{\tau}\int_t^{t+\tau}\left[\frac{1}{F_{u-}} - \frac{1}{F_t}\right]dF_u + \frac{2}{\tau}\sum_{u=N_t}^{N_{t+\tau}}\left[\frac{J_u^2}{2} + J_u + 1 - \exp(J_u)\right]$$

where F_t is the time-t futures price of the underlying asset for maturity $t+\tau$.[5] Using the representation of the so-called "log-contract" in terms of European call and put payoffs gives

$$\left[\frac{F_{t+\tau}}{F_t} - 1 - \log\frac{F_{t+\tau}}{F_t}\right] = \int_0^{F_t}\frac{(K - F_{t+\tau})^+}{K^2}dK + \int_{F_t}^{\infty}\frac{(F_{t+\tau} - K)^+}{K^2}dK.$$

Plugging this expression into $\mathrm{QV}_{t,t+\tau}$ and taking time-t conditional Q expectation of $\mathrm{QV}_{t,t+\tau}$

$$\begin{aligned}
\mathrm{VS}_{t,t+\tau} &= E_t^Q[\mathrm{QV}_{t,t+\tau}] \\
&= \frac{2\exp(r\tau)}{\tau}\int_0^{\infty}\frac{\Theta_t(K,t+\tau)}{K^2}dK + \frac{2}{\tau}E_t^Q\sum_{u=N_t}^{N_{t+\tau}}\left[\frac{J_u^2}{2} + J_u + 1 - \exp(J_u)\right] \\
&= \mathrm{VIX}_{t,t+\tau} + \frac{2}{\tau}E_t^Q\sum_{u=N_t}^{N_{t+\tau}}\left[\frac{J_u^2}{2} + J_u + 1 - \exp(J_u)\right] \qquad (4)
\end{aligned}$$

where $\Theta_t(K, t+\tau)$ is time-t price of an out-of-the-money option with strike K and maturity $t+\tau$, with obvious notation for $\mathrm{VIX}_{t,t+\tau}$. The calculation of the (squared) VIX index is based essentially on the formula $\mathrm{VIX}_{t,t+\tau}$ above and uses European options on the S&P 500 index (SPX), calendar day counting convention, linear interpolation of options whose maturities straddle 30 days; see e.g., Carr and Wu (2006) for a detailed description of the VIX calculation. The key point for our analysis is that the difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ is a model-free measure of the jump component in VS rates. If the jump component is zero, i.e. $J_u = 0$ and/or the intensity of N_t is zero, then $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ is zero as well. If the jump component is not zero and priced, $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ tends to positive. The reason is that the function in the square brackets in (4) is downward sloping and passing through the origin. If the jump distribution under Q is shifted to the left, suggesting jump risk being priced, the last expectation in (4) tends to be positive.

At least two other reasons are conceivable for a non-zero difference of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. The first reasons is that since European options on S&P 500 index (SPX) are likely to be more liquid

[5]Spot and futures prices have the same quadratic variation as the drift becomes negligible when the sampling frequency goes to zero.

than VS contracts, a larger liquidity risk premium could be embedded in VS rates. Long position in VS contracts should induce higher returns when compared to a perfectly liquid VS market. However, this implies that liquidity issues would induce lower VS rates. Thus, if anything, liquidity issues should bias downward zero an otherwise larger positive difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$. A second explanation could be that SPX and VS are segmented or disconnected markets. Then comparing asset prices from the two markets would not provide any valuable information. This is certainly not the case because VS contracts are typically hedged with SPX options.[6]

Following the revised VIX methodology (`http://www.cboe.com/VIX/`) we calculate daily VIX-type indices, $\text{VIX}_{t,t+\tau}$, for $\tau = 2$, 3, and 6 months to maturity from January 4, 1996 to September 2, 2010 and compute the difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$. SPX options are downloaded from OptionMetrics. Although it is straightforward to calculate VIX-type indices for longer maturities, interpolation of existing maturities straddling 12 and 24 months is likely to introduce larger approximation errors. Table 1 shows summary statistics of calculated VIX-type indices. These indices show qualitatively the same term structure features as VS rates. However, on average VS rates are systematically higher, more volatile, skewed, and leptokurtic than VIX-type indices for each maturity. The difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ increases with time to maturity. Figure 2 shows time series plots of $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ for the various time to maturities. Such differences are mostly but not always positive, larger during market turmoils (when jumps in stock price are more likely and investors may care more about jump risk) and can be sizeable also in quite times. Hence a positive difference is not only a crisis phenomenon. These findings are consistent with the presence of a jump component embedded in VS rates.

2.3. *Stochastic Volatility Model*

Now we parameterize model (1). Under P the stock price has the following dynamic

$$\begin{aligned} dS_t/S_{t_-} &= \mu_t dt + \sqrt{(1-\rho^2)v_t}\, dW_{1t}^P + \rho\sqrt{v_t}\, dW_{2t}^P + (\exp(J_t^P) - 1)\, dN_t - \nu_t^P\, dt \\ dv_t &= k_v^P (m_t\, k_v^Q / k_v^P - v_t)dt + \sigma_v \sqrt{v_t}\, dW_{2t}^P \\ dm_t &= k_m^P(\theta_m^P - m_t)dt + \sigma_m \sqrt{m_t}\, dW_{3t}^P \end{aligned} \tag{5}$$

[6] The difficulties of carrying out this hedging even featured in The Wall Street Journal in October 2008 when volatility was at historically high values; see Schultes (2008).

where $\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t$, all Brownian increments, $dW_{it}^P, i = 1,2,3$, are uncorrelated,[7] and ρ is the instantaneous correlation between stock returns and spot variance changes. The random jump size J_t^P is independent of the filtration generated by the Brownian motions and jump process, and normally distributed with mean μ_j^P and variance σ_j^2, hence $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$ is the Laplace transform of the random jump size. Similarly $g^Q = \exp(\mu_j^Q + \sigma_j^2/2) - 1$. The jump intensity is $\lambda_t = \lambda_0 + \lambda_1 v_t$, where λ_0 and λ_1 are positive constants. This specification allows for more price jumps to occur during more volatile periods with the intensity bounded away from 0 by λ_0. Bates (2000), Pan (2002), Eraker (2004), Broadie, Chernov, and Johannes (2007), among many others, assume normally distributed jump prices and provide empirical evidence that jumps in stock returns are more likely to occur when volatility is high, supporting model specification (5). Using alternative approaches Aït-Sahalia (2002b) and Carr and Wu (2003), among others, provide additional evidence for jumps in stock returns. The spot variance, v_t, follows a two-factor model where $m_t\, k_v^Q/k_v^P$ is its stochastic long-run mean or central tendency. The speed of mean reversion is k_v^P under P, k_v^Q under Q and $k_v^P = k_v^Q - \gamma_2\sigma_v$ where γ_2 is the market price of risk for W_{2t}^P; Section 2.4 motivates the last equality. The stochastic long run mean of v_t is controlled by m_t which follows its own stochastic mean reverting process and mean reverts to a positive constant θ_m^P when the speed of mean reversion k_m^P is positive. Typically, v_t is fast mean reverting and volatile to capture sudden movements in volatility, while m_t is more persistent and less volatile to capture long run movements in volatility. The square-root specification of the diffusion components $\sigma_v\sqrt{v_t}$ and $\sigma_m\sqrt{m_t}$ is adopted to keep model (5) close to commonly used models; see e.g., Chernov and Ghysels (2000), Pan (2002), Broadie, Chernov, and Johannes (2007), and Todorov (2010).[8]

Under Q the ex-dividend price process evolves as

$$\begin{aligned} dS_t/S_{t_-} &= (r-\delta)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^Q + \rho\sqrt{v_t}\,dW_{2t}^Q + (\exp(J_t^Q)-1)\,dN_t^Q - \nu_t^Q\,dt \\ dv_t &= k_v^Q(m_t - v_t)dt + \sigma_v\sqrt{v_t}\,dW_{2t}^Q \\ dm_t &= k_m^Q(\theta_m^Q - m_t)dt + \sigma_m\sqrt{m_t}\,dW_{3t}^Q \end{aligned} \tag{6}$$

where r is the risk-free rate, δ the dividend yield (both taken to be constant for simplicity only),

[7]Note that $\sqrt{(1-\rho^2)}\,dW_{1t}^P + \rho\,dW_{2t}^P = d\tilde{W}_t^P$ in model (1).

[8]The analyses undertaken below, except the one for the integrated equity risk premium, are based on analytical expressions which do not require such square-root specification For instance $\sqrt{v_t}$ could be replaced by $v_t^{\gamma_v}$ with γ_v being one more parameter to be estimated. The level of variance swap rates is not affected by such alternative specification, but of course the dynamic is.

and Brownian motions $W_i^Q, i = 1, 2, 3$, jump size J^Q, jump process N^Q, and its compensator ν^Q are governed by the measure Q. When no confusion arises superscripts P and Q are omitted.

Given the parametric model above, the VS rate can easily be calculated. In (3), interchanging expectation and integration (justified by Tonelli's theorem), and exploiting independence between J^Q and N^Q

$$\begin{aligned} \mathrm{VS}_{t,t+\tau} &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q[v_s]ds + \frac{1}{\tau}E^Q[J^2]\,E_t^Q[N_{t+\tau} - N_t] \\ &= E^Q[J^2]\lambda_0 + (1 + \lambda_1 E^Q[J^2])[(1 - \phi_v^Q(\tau) - \phi_m^Q(\tau))\theta_m^Q + \phi_v^Q(\tau)v_t + \phi_m^Q(\tau)m_t] \quad (7) \end{aligned}$$

where $E^Q[J^2] = E_t^Q[J^2]$ as the random jump size is time-homogeneous, and

$$\begin{aligned} \phi_v^Q(\tau) &= (1 - \exp(-k_v^Q\tau))/(k_v^Q\tau) \\ \phi_m^Q(\tau) &= \Big(1 + \exp(-k_v^Q\tau)k_m^Q/(k_v^Q - k_m^Q) - \exp(-k_m^Q\tau)k_v^Q/(k_v^Q - k_m^Q)\Big)/(k_m^Q\tau). \end{aligned}$$

Given the linearity of the variance swap payoff in the spot variance, only the drift of v_t enters the variance swap rate. The diffusion part of v_t (or volatility of volatility) affects only the dynamic of $\mathrm{VS}_{t,t+\tau}$.[9] The Q-expectation of squared jump size, $E^Q[J^2]$, provides a constant contribution to the variance swap rate (independent of the time to maturity), while the stochastic intensity provides a time-varying contribution to $\mathrm{VS}_{t,t+\tau}$ given by $E_t^Q[N_{t+\tau} - N_t]$. Compared to a no-jump model, e.g., $\lambda_0 = \lambda_1 = 0$, the jump component shifts the term structure of VS rates upward.

Besides the empirical evidence on jumps in stock returns, the main motivation for introducing such a jump component is to account for a jump component in VS rates as documented by our model-free analysis in Section 2.2. The two-factor model for the spot variance is key to reproduce the variety of shapes of VS term structure described in Section 2.1. Indeed, according to our model estimates, when $\tau \to 0$, $\phi_v^Q(\tau) \to 1$ and $\phi_m^Q(\tau) \to 0$ hence short maturities VS rates are mainly determined by v_t. When $\tau \to \infty$, $\phi_v^Q(\tau) \to 0$ and $\phi_m^Q(\tau) \to 0$ hence long maturities VS rates are mainly determined by θ_v^Q. As $\phi_m^Q(\tau)$ is slower than $\phi_v^Q(\tau)$ in approaching zero, m_t has also a relatively large impact on long maturity VS rates. In Equation (7) the last term in square brackets is a weighted average of θ_m^Q, v_t and m_t. The relative level of the three components controls the shape of the term structure. For example $\mathrm{VS}_{t,t+\tau}$ is monotonically increasing in τ when $v_t < m_t = \theta_m^Q$, or the term structure is hump-shape when $v_t < m_t$ and $m_t > \theta_m^Q$.

[9] This can easily be seen by specifying some dynamic for the spot variance, such as the two-factor model in Equation (6), calculating $\int_t^{t+\tau} E_t^Q[v_s]ds$ explicitly, i.e. the non-annualized VS rate, and then applying Itô's formula to it.

Moreover, the two-factor model is consistent with the different persistence, volatility and higher order moment features of VS rates observed empirically. According to our estimates, v_t is for example less persistence, more volatile and positively skewed than m_t. Model-based short maturities VS rates inherit such features when compared to long maturities rates. The empirical Section 4 shows that Model (5)–(6) matches such features quite well. As shown in Section 2.1 two principal components virtually explain all the variation in VS rates. Thus PCA supports the two-factor model as well. All in all, Model (5)–(6) appears to be the most parsimonious parametric model consistent with the model-free analysis of actual VS rates.

Imposing the restriction $m_t = \theta_v^Q$ for all t and $\lambda_0 = \lambda_1 = 0$ implies that the variance dynamic in Equation (6) follows the Heston model. The VS rate becomes $\mathrm{VS}_{t,t+\tau} = (1 - \phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)v_t$, i.e. a weighted average of v_t and θ_v^Q. Hence the term structure of VS rates can only be upward or downward sloping at each point in time, depending on whether $v_t < \theta_v^Q$ or $v_t > \theta_v^Q$, respectively. Moreover the persistence of VS rates is the same for all maturities as only one factor, v_t, is driving all VS rates. These features of model-based VS rates are contrast with the empirical features of actual VS rates. Egloff, Leippold, and Wu (2010) study VS rates and optimal allocations in these contracts under the one- and two-factor stochastic volatility model when the jump component is absent, e.g., $\lambda_0 = \lambda_1 = 0$.

Using stock and option prices, previous studies have documented that two factors are necessary to describe stochastic volatility dynamics; see e.g., Andersen, Benzoni, and Lund (2002), Alizadeh, Brandt, and Diebold (2002) and Chernov, Gallant, Ghysels, and Tauchen (2003), and Todorov (2010). As in Model (5)–(6) the two factors operate at two different time scales, i.e. one factor is fast mean reverting and volatile while the other factor is more persistent and less volatile. Specifically, in particular for option pricing purposes, the preferred specification for the spot variance dynamic is of the following form

$$dv_t = k_v^Q(\theta_v^Q - v_t)dt + \sigma_v \sqrt{v_t}\, dW_{2t}^Q + J_t^v\, dN_t^v \tag{8}$$

where J_t^v is a time-homogenous positive random jump size, typically exponentially distributed, and N_t^v is a counting process of volatility jumps, often with constant intensity λ^v, independent of J_t^v; see e.g., Eraker, Johannes, and Polson (2003), Eraker (2004) and Broadie, Chernov, and Johannes (2007). The jump component is the fast moving factor and its main contribution is to generate enough skewness in short maturity implied volatility smiles. Unfortunately, this kind of models cannot generate the variety of shapes of VS term structures observed empirically. The fundamental reason is that the volatility jump is not an autonomous state variable. Introducing a state dependent jump component, for e.g., when the jump intensity is an affine function of the spot variance, it does not alter the conclusion. Indeed, the VS rate based on Equation (8) is exactly the same as in the Heston model when replacing θ_v^Q by $(\theta_v^Q + E^Q[J^v]\,\lambda^v/k_v^Q)$, and assuming no jump component in stock returns.[10] Thus Model (8) shares the same drawbacks as one-factor models.

2.4. Market Price of Risks

As in Pan (2002), Aït-Sahalia and Kimmel (2007), and others, we specify the market price of risks for the Brownian motions as

$$\Lambda_t = [\gamma_1\sqrt{(1-\rho^2)v_t}, \quad \gamma_2\sqrt{v_t}, \quad \gamma_3\sqrt{m_t}]'$$

where $'$ denotes transpose. Thus, P and Q parameters controlling v_t and m_t are related as

$$k_v^P = k_v^Q - \gamma_2\sigma_v, \quad k_m^P = k_m^Q - \gamma_3\sigma_m, \quad \theta_m^P = \theta_m^Q\, k_m^Q/k_m^P.$$

The jump size risk premium is $(g^P - g^Q) = \exp(\mu_j^P + \sigma_j^2/2) - \exp(\mu_j^Q + \sigma_j^2/2)$. Note that the variance of the jump size is the same under P to Q. Although this constraint could be relaxed without introducing arbitrage opportunities, it is implied for e.g., by the general equilibrium model in Naik and Lee (1990).[11] As e.g., in Pan (2002), Eraker (2004), Broadie, Chernov, and Johannes (2007), we assume that the jump intensity is the same under both measures. This assumption implies that all the jump risk premium is absorbed by the jump size risk premium, $(g^P - g^Q)$. The total jump risk premium is time-varying and given by $(g^P - g^Q)(\lambda_0 + \lambda_1 v_t)$. The main motivation for this assumption is the well-known limited ability of estimating jump components in stock returns and the corresponding risk premium using daily data. It is so because these jumps are rare events. According to our estimates they occur roughly four times a year and previous studies (e.g., Broadie, Chernov, and Johannes (2007)) reported somehow smaller frequencies. Even using 15 years of daily data, accurate estimation of risk premiums for both jump-size and jump-timing appears to be challenging. Nevertheless later on we will relax this assumption.

[10] When there is no jump component in stock returns, the VS rate is $\mathrm{VS}_{t,t+\tau} = \tau^{-1}\int_t^{t+\tau} E_t^Q[v_s]\,ds$ where $E_t^Q[v_s] = e^{-k_v^Q(s-t)}v_t + \int_t^s e^{-k_v^Q(s-u)}(k_v^Q\theta_v^Q + E^Q[J^v]\lambda^v)\,du$. Adding a positive jump component as in Model (5)–(6) would simply shift this VS rate upward.

[11] The equilibrium model is the Lucas economy with power utility over consumption or wealth. While the assumptions behind the model are reasonable, Broadie, Chernov, and Johannes (2007) provide evidence that relaxing the constraint $\mathrm{Var}^P[J] = \mathrm{Var}^Q[J]$ improves options fitting.

The jump component in the stock price makes the market incomplete with respect to the risk-free bank account, the stock and any finite number of derivatives. Hence, the state price density is not unique. One of such specifications is

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \Lambda_s' \, dW_s^P - \frac{1}{2}\int_0^t \Lambda_s'\Lambda_s \, ds\right) \prod_{u=1}^{N_t} \exp\left(\frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2} J_u\right). \quad (9)$$

Appendix A shows that Equation (9) is a valid state price density. The first exponential function is the usual Girsanov change of measure of the Brownian motions and the second term is the change of measure for the jump component. It has a similar expression as the change of measure for the Brownian motions because the jump size is normally distributed. As shown in Equation (9), in the economy described by this model, jumps are priced because when a jump occurs the state price density jumps as well. When $\mu_j^Q < \mu_j^P$ and a negative jump occurs ($J_u < 0$) the state price density jumps up giving "high" prices to (Arrow–Debreu) securities with payoffs in bad states of the economy. In our empirical estimates we invariably find that $\mu_j^Q < \mu_j^P$.

3. Estimation Method

Model (5)–(6) is estimated by merging the approaches in Aït-Sahalia and Kimmel (2007, 2010). The procedure combines time series information in stock returns and cross sectional information in term structures of VS rates in the same spirit as e.g., Chernov and Ghysels (2000) and Pan (2002). Hence P and Q parameters, including risk premiums, are estimated simultaneously exploiting internal consistency of the model and making inference procedure theoretically sounded and more accurate.

Let $X_t' = [\log(S_t), Y_t']$ denote the state vector where $Y_t = [v_t, m_t]'$. The spot variance and its stochastic long run mean are not observed and will be extracted from observed VS rates. Likelihood-based estimation requires evaluation of the likelihood function of stock returns and term structures of variance swap rates for each parameter vector during a likelihood search. The procedure for evaluating the likelihood function consists of four steps. First, we extract the unobserved state vector Y_t from a set of benchmark variance swap rates, assumed to be observed without error. Second, we evaluate the joint likelihood of the stock returns and extracted time series of latent states, using an approximation to the likelihood function. Third, we multiply this joint likelihood by a Jacobian determinant to compute the likelihood of observed data, namely stock returns and term structures of VS rates. Finally, for the remaining VS rates assumed to be observed with error, we calculate the likelihood of the observation errors induced by the previously extracted state variables. The product of the two likelihoods gives the joint likelihood of the term structures of all variance swap rates and stock returns. This procedure is repeated for different values of the parameter vector until the maximum of the likelihood function is found.

3.1. Extracting State Variables from Variance Swap Rates

Equation (6) implies that variance swap rates are affine in the unobserved state variables. This feature suggests the following natural procedure to extract latent states and motivates our likelihood-based approach. The unobserved part in the state vector, Y_t, is ℓ dimensional, where $\ell = 2$ in Model (5)–(6). As the method can be applied for any $\ell \geq 1$, provided of course that enough data are available, we describe the procedure for a generic ℓ. Suppose that ℓ variance swap rates are observed without error. Thus, at each day t, the state vector Y_t is exactly identified by the ℓ variance swap rates. Denote by $\tau_1, \ldots, \tau_\ell$ their times to maturity. The observed ℓ variance swap rates, $\mathrm{VS}_{t,t+\tau_1}, \ldots, \mathrm{VS}_{t,t+\tau_\ell}$ jointly follow a Markov process and satisfy

$$\begin{bmatrix} \mathrm{VS}_{t,t+\tau_1} \\ \vdots \\ \mathrm{VS}_{t,t+\tau_\ell} \end{bmatrix} = \begin{bmatrix} a(\tau_1;\Theta) \\ \vdots \\ a(\tau_\ell;\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_1;\Theta)' \\ \vdots \\ b(\tau_\ell;\Theta)' \end{bmatrix} \begin{bmatrix} Y_{t1} \\ \vdots \\ Y_{t\ell} \end{bmatrix}.$$

The previous equation in matrix form reads $\mathrm{VS}_{t,\cdot} = a(\Theta) + b(\Theta)Y_t$, where $a(\Theta)$ is the $(\ell \times 1)$ vector and $b(\Theta)$ the $(\ell \times \ell)$ matrix. The current value of the unobserved state vector Y_t can easily be found by solving the equation for Y_t, i.e. $Y_t = b(\Theta)^{-1}[\mathrm{VS}_{t,\cdot} - a(\Theta)]$. For example rearranging Equation (7) gives $\mathrm{VS}_{t,t+\tau} = a(\tau;\Theta) + b(\tau;\Theta)'[v_t, m_t]'$, where

$$\begin{aligned} a(\tau;\Theta) &= E^Q[J^2]\lambda_0 + (1 + \lambda_1 E^Q[J^2])(1 - \phi_v^Q(\tau) - \phi_m^Q(\tau))\theta_m^Q \\ b(\tau;\Theta)' &= (1 + \lambda_1 E^Q[J^2])\ [\phi_v^Q(\tau), \quad \phi_m^Q(\tau)]. \end{aligned}$$

The affine relation between VS rates and latent variables makes recovering the latter nearly costless when compare to recovering them from, say, standard call and put options as for e.g., in Pan (2002) and Aït-Sahalia and Kimmel (2007).

3.2. Likelihood of Stock Returns and Variance Swap Rates Observed Without Error

The extracted time series values of the unobserved state vector Y_t at dates $t_0, t_1, \ldots, t_n$ allows to infer the dynamics of the state variables $X_t' = [\log(S_t), Y_t']$ under the objective probability P.

Since the relationship between the unobserved state vector Y_t and variance swap rates is affine, the transition density of variance swap rates can be derived from the transition density of Y_t by a change of variables and multiplication by a Jacobian determinant which depends, in this setting, on model parameters but not on the state vector. Let $p_X(x_\Delta|x_0;\Theta)$ denote the transition density of the state vector X_t under the measure P, i.e. the conditional density of $X_{t+\Delta} = x$ given $X_t = x_0$. Let $A_t = [\log(S_t), \mathrm{VS}_{t,t+\tau_1}, \ldots, \mathrm{VS}_{t,t+\tau_\ell}]'$ be the vector of observed asset prices and $p_A(a|a_0;\Theta)$ the corresponding transition density. Observed asset prices, A_t, are given by an affine transformation of X_t

$$A_t = \begin{bmatrix} \log(S_t) \\ \mathrm{VS}_{t,\cdot} \end{bmatrix} = \begin{bmatrix} \log(S_t) \\ a(\Theta) + b(\Theta)Y_t \end{bmatrix} = \begin{bmatrix} 0 \\ a(\Theta) \end{bmatrix} + \begin{bmatrix} 1 & 0' \\ 0 & b(\Theta) \end{bmatrix} X_t$$

and rewritten in matrix form reads $A_t = \tilde{a}(\Theta) + \tilde{b}(\Theta)X_t$, with the obvious notation. The Jacobian term of the transformation from X_t to A_t is therefore

$$\det\left|\frac{\partial A_t}{\partial X_t'}\right| = \det\left|\tilde{b}(\Theta)\right| = \det\left|b(\Theta)\right|.$$

In Model (5)–(6), $\det|b(\Theta)| = (1 + \lambda_1 E^Q[J^2])^2\ (\phi_v^Q(\tau_1)\phi_m^Q(\tau_2) - \phi_v^Q(\tau_2)\phi_m^Q(\tau_1))$. Since $X = \tilde{b}(\Theta)^{-1}[A - \tilde{a}(\Theta)]$,

$$p_A(A|A_0;\Theta) = \det\left|b(\Theta)^{-1}\right|\ p_X(\tilde{b}(\Theta)^{-1}[A - \tilde{a}(\Theta)]|\tilde{b}(\Theta)^{-1}[A_0 - \tilde{a}(\Theta)];\Theta). \qquad (10)$$

As the vector of asset prices is Markovian, applying Bayes' Rule, the log-likelihood function of the asset price vector A_t sampled at dates $t_0, t_1, \ldots, t_n$ has the simple form

$$l_n(\Theta) = \sum_{i=1}^{n} l_A(A_{t_i}|A_{t_{i-1}};\Theta) \qquad (11)$$

where $l_A = \ln p_A$. As usual in likelihood estimation, we discard the unconditional distribution of the first observation since it is asymptotically irrelevant.

In our applications and Monte Carlo simulation below, models are estimated using daily data, hence the sampling process is deterministic and $t_i - t_{i-1} = \Delta = 1/252$; see Aït-Sahalia and Mykland (2003) for a treatment of maximum likelihood estimation in the case of randomly spaced sampling times.

3.3. *Likelihood of Stock Returns and All Variance Swap Rates*

From the coefficients $a(\tau;\Theta)$ and $b(\tau;\Theta)$ and the values of the state vector X_t found in the first step, we can calculate the implied values of the variance swap rates which are assumed to be observed with error and whose time to maturities are denoted by $\tau_{\ell+1}, \ldots, \tau_{\ell+h}$

$$\begin{bmatrix} \mathrm{VS}_{t,t+\tau_{\ell+1}} \\ \vdots \\ \mathrm{VS}_{t,t+\tau_{\ell+h}} \end{bmatrix} = \begin{bmatrix} a(\tau_{\ell+1};\Theta) \\ \vdots \\ a(\tau_{\ell+h};\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_{\ell+1};\Theta)' \\ \vdots \\ b(\tau_{\ell+h};\Theta)' \end{bmatrix} \begin{bmatrix} Y_{t1} \\ \vdots \\ Y_{t\ell} \end{bmatrix}.$$

The observation errors, denoted by $\varepsilon(t, \tau_{\ell+i})$, $i = 1, \ldots, h$, are the differences between such model-based implied VS rates and actual VS rates from the data. By assumption, these errors are Gaussian with zero mean and constant variance, independent of the state process and across time, but possibly correlated across maturities. The joint likelihood of the observation errors can be calculated from the h dimensional Gaussian density function. Since the observation errors are independent of the state variable process, the joint likelihood of stock returns and all observed variance swap rates is simply the product of the likelihood of stock returns and variance swap rates observed without error, multiplied by the likelihood of the observation errors. Equivalently, the two log-likelihoods can simply be added to obtain the joint log-likelihood of stock returns and all variance swap rates.

3.4. Likelihood Approximation

Since the state vector X is a continuous-time multivariate jump diffusion process, its transition density is usually unknown. Since jumps in stock returns are rare events, it is very unlikely that more than one jump occurs on a single day Δ. This observation motivates the following Bayes' approximation of p_X

$$p_X(x_\Delta|x_0) = p_X(x_\Delta|x_0, N_\Delta = 0)\,\Pr(N_\Delta = 0) + p_X(x_\Delta|x_0, N_\Delta = 1)\,\Pr(N_\Delta = 1) + o(\Delta)$$

where $\Pr(N_\Delta = j)$ is the probability that j jumps occur at day Δ, omitting the dependence on Θ for brevity. In Model (5)–(6), the largest contribution to the transition density of X (hence to the likelihood) comes from the first term when conditioning on no jump occurring during Δ, i.e. $N_\Delta = 0$. The probability of such event, $\Pr(N_\Delta = 0)$, is large and of the order $1 - (\lambda_0 + \lambda_1 v_0)\,\Delta$. The contribution of the second term is only of the order $(\lambda_0 + \lambda_1 v_0)\,\Delta$. As in our setting Δ is one day, the contribution of higher order terms appear to be quite modest. The advantage of this approximation is that the leading term $p_X(x_\Delta|x_0, N_\Delta = 0)$ can be accurately approximated by the likelihood expansion method. Aït-Sahalia (1999, 2002b) introduces such density approximation in closed-form in the univariate setting and Aït-Sahalia (2008) extends the method to multivariate diffusions. We refer the reader to these studies for the description of the method. Below we only summarize the main idea. The expansion for the transition density of X conditioning on no jump has the form of a Taylor series in Δ at order K

$$p^{(K)}(x|x_0;\Theta) = \Delta^{-(\ell+1)/2} \exp\left[-\frac{C^{(-1)}(x|x_0;\Theta)}{\Delta}\right] \sum_{k=0}^{K} C^{(k)}(x|x_0;\Theta)\frac{\Delta^k}{k!}. \tag{12}$$

The series can be calculated up to arbitrary order K and the unknowns are the coefficients $C^{(k)}$ corresponding to each Δ^k, $k = -1, 0, \ldots, K$. In the present setting the continuous part of the dynamic X is generally not reducible (Aït-Sahalia, 2008). The approach is then to expand each coefficient $C^{(k)}$ in a Taylor series in $(x - x_0)$ at order $j_k = 2(K - k)$. Denoting $C^{(j_k,k)}$ such expansions, the transition density expansion is

$$\tilde{p}^{(K)}(x|x_0;\Theta) = \Delta^{-(\ell+1)/2} \exp\left[-\frac{C^{(j_{-1},-1)}(x|x_0;\Theta)}{\Delta}\right] \sum_{k=0}^{K} C^{(j_k,k)}(x|x_0;\theta)\frac{\Delta^k}{k!}. \tag{13}$$

Coefficients $C^{(j_k,k)}$ are computed by forcing the expansion (13) to satisfy, to order Δ^K, the forward and backward Kolmogorov equations. A key feature of the method is that the coefficients are obtained in closed-form by solving a system of linear equations. This holds true for arbitrary specifications of the dynamics of the state vector X. Moreover, the coefficients need to be computed only once and not at each iteration of the likelihood search. The density in Equation (13) provides a virtually exact approximation of the transition density of X; see e.g., Jensen and Poulsen (2002). In our simulation and empirical applications we use expansions at order $K = 2$ which provide already highly accurate approximation to the unknown density.[12]

3.5. *Monte Carlo Simulation*

We run a Monte Carlo simulation to check the accuracy of the previous estimation method. Model (5)–(6) is simulated at an intraday frequency of 15 minutes. Using an Euler discretization we simulate a long trajectory of S, v and m. Then, we pick one value every 30 values to obtain a daily sample. VS rates are calculated for 3- and 12-month to maturity using the simulated values. Each sample consists of N = 10,000 values of the underlying stock price and VS rates. Each simulated path starts with variance and its stochastic long run mean at their unconditional means and the stock price at 100. To reduce the impact of such initial values on the simulated trajectory, an initial 500 values are generated and then discarded, taking the last one as the starting point for the sample trajectory. Model (5)–(6) is then estimated using each

[12] Yu (2007) studies transition density expansions for jump-diffusion models but we do not purse this direction further.

simulated path. This procedure is repeated 900 times. Table 2 reports the simulation results and shows that the estimation procedure is generally accurate. As expected, notoriously challenging parameters, such as the risk premium γ_1 or the affine jump intensity λ_1, are estimated less precisely.

4. Fitting Variance Swap Rates

Table 3 reports parameter estimates for Model (5)–(6). Overall, the model provides a good fit to VS rates and S&P 500 returns. The spot variance is relatively fast mean reverting as k_v^P implies a half-life[13] of 36 days. Its stochastic long run mean is slowly mean reverting with a half-life of almost 3 years. The volatility of v_t is almost 3 times that of m_t. The instantaneous correlation between stock returns and variance changes, ρ, is -71%, confirming the so-called leverage effect. The long-run average volatility, $\sqrt{\theta_m^P}$, is 21%. Both γ_2 and γ_3 are negative implying negative variance risk premium. The correlation parameter for pricing error terms, ρ_e, is slightly negative[14] suggesting that the model does not produce any systematic pricing error. By contrast, this correlation parameter in other nested models, such as the Heston model, turn out to be positive which implies that variance swaps are systematically underpriced or overpriced in nested models.

The expected jump size is positive under the objective probability measure, μ_j^P, and slightly negative under the risk neutral measure, μ_j^Q, suggesting a positive jump risk premium. Estimates of jump intensity implies on average 6 jumps per year, a somewhat larger number than what reported in previous studies.

Table 4 shows the pricing errors of Model (5)–(6) when fitting VS rates. For comparison, pricing errors of the Heston model are reported as well. The pricing error is defined as the model-based VS rate minus actual VS rate. Model (5)–(6) fits VS rates well both in- and out-sample and significantly outperforms the Heston model. For example, its root mean square error is 6 times smaller than that of the Heston model when fitting 24-month to maturity VS rates. As pricing errors are small, Model (5)–(6) captures well all empirical features of VS rates documented in Table 1.

[13] The half-life is defined as the time necessary to halve a unit shock and is given by $-\log(0.5)/k_v^P \times 252$ in number of days.

[14] The determinant of the 3×3 error term correlation matrix is $2\rho_e^3 - 3\rho_e^2 + 1$, which is strictly positive when $\rho_e > -0.5$.

5. Risk Premiums

5.1. *Instantaneous Risk Premiums*

Model (5)–(6) features four instantaneous risk premiums: Diffusive Risk Premium (DRP), Jump Risk Premium (JRP), Variance Risk Premium (VRP), and Long-run Mean Risk Premium (LRMRP) which are defined as

$$
\begin{aligned}
\mathrm{DRP}_t &= (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t \\
\mathrm{JRP}_t &= (g^P - g^Q)(\lambda_0+\lambda_1 v_t) \\
\mathrm{VRP}_t &= \gamma_2\sigma_v v_t \\
\mathrm{LRMRP}_t &= \gamma_3\sigma_m m_t
\end{aligned}
$$

DRP is the remuneration for diffusive-type risk only (due to Brownian motions driving stock prices), while JRP for the jump component in stock price. The Equity Risk Premium (ERP) is the sum of the two, i.e. $\mathrm{ERP}_t = \mathrm{DRP}_t + \mathrm{JRP}_t$. Under P the instantaneous mean growth rates of v_t and m_t are different than under Q, and the differences are given by VRP_t and LRMRP_t, respectively. As γ_2 and γ_3 are estimated to be negative, v_t and m_t are higher under Q than under P on average, and VRP and LRMRP are both negative. Table 5 shows estimated risk premiums. During our in-sample period, January 1996 to April 2007, the average ERP is 6.8%, 5.5% of which is due to the JRP. Thus, the jump risk premium accounts for the largest fraction of the equity risk premium. Jump prices are rare events, but arguably jump risk is important as it cannot be hedged with any finite number of securities. The average VRP is also substantial and around -3.4%, while the LRMRP is much lower and around -0.4%. During the out-sample period, April 2007 to September 2010, all risk premiums almost doubled reflecting the unprecedented turmoil in financial markets around Lehman Brothers bankruptcy.

5.2. *Integrated Risk Premiums*

5.2.1 Integrated Equity Risk Premium

The annualized integrated Equity Risk Premium (IERP) is defined as

$$
\begin{aligned}
\mathrm{IERP}_{t,t+\tau} &= E_t^P[(S_{t+\tau}-S_t)/S_t]/\tau - E_t^Q[(S_{t+\tau}-S_t)/S_t]/\tau \\
&= E_t^P[e^{\int_t^{t+\tau}\mu_s\,ds}]/\tau - e^{(r-\delta)\tau}/\tau
\end{aligned}
$$

and represents the expected excess return from buying and holding the S&P 500 index from t to $t+\tau$. The IERP can be decomposed in the continuous and jump part, i.e., $\mathrm{IERP}_{t,t+\tau} =$

$\text{IERP}^c_{t,t+\tau} + \text{IERP}^j_{t,t+\tau}$, where

$$\begin{aligned}\text{IERP}^c_{t,t+\tau} &= E^P_t[e^{\int_t^{t+\tau}(r-\delta+\gamma_1(1-\rho^2)v_s+\gamma_2\rho v_s)\,ds}]/\tau - e^{(r-\delta)\tau}/\tau \\ \text{IERP}^j_{t,t+\tau} &= E^P_t[e^{\int_t^{t+\tau}(g^P-g^Q)(\lambda_0+\lambda_1 v_s)\,ds}]/\tau.\end{aligned}$$

This decomposition allows to quantify how the various risks contribute to the IERP. The corresponding P-expectations can be computed analytically using the transform analysis in Duffie, Pan, and Singleton (2000).[15] The analytical expressions are of the form $\exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$, where $A(\tau)$, $B(\tau)$ and $C(\tau)$ are positive coefficients. Thus, in quite times, when the spot variance v_t and its stochastic long run mean m_t are low, IERPs are low as well. When the market crashes and v_t and m_t increase significantly, IERPs are large, reflecting the stressed asset prices and the large risk premium an investor might expect to earn by going long in the S&P 500 precisely during those difficult times.

In contrast to instantaneous risk premiums, at each time t, the integrated equity risk premium depends on a time horizon τ. Such a term structure of the IERP provides information about future asset price dynamics. For example, suppose that the term structure is downward sloping and the 2-month IERP is significantly larger than IERP for longer horizons, as it was the case when Lehman Brothers collapsed. This suggests that investing for a 2-month period during those difficult times would have generated, on average, a larger annualized return than investing for longer periods, as the market expected a fast rebound of prices, in relative terms.

Table 6 reports statistics for the integrated equity risk premium over 2-, 6-, 12- and 24-month horizon.[16] The term structure is on average upward sloping from January 1996 to April 2007, our in-sample period, and about 7%. From April 2007 to September 2010, the IERP is significantly larger and about 9.5%, with an average term structure slightly U-shaped, reflecting the stressed asset prices and high uncertainly around Lehman Brothers bankruptcy. Figure 3 shows the evolution of the IERP over time, along with the S&P 500 index. At the end of 2008

[15] The expectation to be computed is of the form $E^P_t[\exp(K\int_t^{t+\tau} v_s\,ds)]$, where K is a given constant. To calculate it explicitly, define $\psi_t = E^P_t[\exp(K\int_0^T v_s\,ds)]$, which is a martingale by construction for all $t \geq 0$. Guess the following functional form $\psi_t = \exp(K\int_0^t v_s\,ds)\,\exp(A(\tau)+B(\tau)v_t+C(\tau)m_t)$, which is exponentially affine in the state variables and $\tau = T-t$. Applying Itô's lemma to ψ_t and setting its drift to zero, as ψ_t is a P-martingale, give the following differential equations that $A(\tau)$, $B(\tau)$ and $C(\tau)$ must satisfy: $-\partial_\tau A(\tau) + C(\tau)k^P_m\theta^P_m = 0$, $K - \partial_\tau B(\tau) - B(\tau)k^P_v + 0.5\,B(\tau)^2\sigma_v^2 = 0$, $-\partial_\tau C(\tau) + B(\tau)k^Q_v - C(\tau)k^P_m + 0.5\,C(\tau)^2\sigma_m^2 = 0$, where ∂_τ denotes the derivative with respect to τ, with terminal conditions $A(\tau) = B(\tau) = C(\tau) = 0$. As the system is time-homogenous, for each time horizon τ those coefficients need to be computed only once. Thus, at each time t, $E^P_t[\exp(K\int_t^{t+\tau} v_s\,ds)] = \exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$.

[16] As the IERP for the 2- and 3-month horizon are quite close, the latter is not reported in the table.

and beginning of 2009, the term structure of the IERP is downward sloping with the 2-month IERP significantly above 30%. These estimates match well the evolution of the S&P 500 index during that period. From mid-September to mid-November 2008, the S&P 500 index dropped from 1,200 to 750, loosing 37% of its value in two months.

Table 6 also shows that the jump component $\text{IERP}^j_{t,t+\tau}$ contributes significantly more than the diffusive component $\text{IERP}^c_{t,t+\tau}$ to the term structure of the IERP. For example, during our in-sample period, the one-year IERP is 7.3%, but 6% is due to jump risk.

An advantage of studying the term structure of IERP in a parametric model is that such risk premiums and their decompositions are exact. In other words, there is no need of using interpolations or moving average schemes to reduce impact on risk premiums of systematically time varying maturities, as, for e.g., in Bollerslev and Todorov (2011).

5.2.2 Integrated Variance Risk Premium

The annualized integrated variance risk premium (IVRP) is defined as $\text{IVRP}_{t,t+\tau} = E^P_t[\text{QV}_{t,t+\tau}] - E^Q_t[\text{QV}_{t,t+\tau}]$ and represents the expected profit to the long side of a VS contract, which is entered at time t and held till maturity $t+\tau$. Table 6 reports summary statistics of the integrated variance risk premiums and Figure 4 shows the dynamic over time. IVRPs are economically large. For instance, average IVRP for 24-month maturity is -1.7%, during our out-sample period, and can be as large as -5% in variance units. These are large risk premiums compared to an average spot variance of 4% in variance units. In contrast to IVRP based on ex-post realized variance, as e.g., in Carr and Wu (2009), ex-ante, model-based, IVRP is always negative.

The longer the time to maturity the higher in absolute value the annualized IVRP. Thus, the term structure of IVRP is downward sloping suggesting that long maturity VS contracts carry more remuneration for stochastic variance. Indeed, Filipović, Gourier, and Mancini (2011) study optimal investment in VS and show that an optimal strategy is to go short in long maturity VS and partially hedge the exposure by going long in short maturity VS and the underlying stock.[17]

As the quadratic variation can naturally be decomposed in the continuous, $\text{QV}^c_{t,t+\tau}$, and

[17]Egloff, Leippold, and Wu (2010) also study optimal investment in VS, but they reach exactly the opposite conclusion, which can be explained by the different model and market price of risks specifications used in the two studies.

discontinuous, $\mathrm{QV}^j_{t,t+\tau}$, part, i.e.,

$$\mathrm{QV}_{t,t+\tau} = \frac{1}{\tau}\int_t^{t+\tau} v_s\,ds + \frac{1}{\tau}\sum_{u=N_t}^{N_{t+\tau}} J_u^2 = \mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau}$$

the IVRP can also be decomposed as

$$\begin{aligned}
\mathrm{IVRP}_{t,t+\tau} &= E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}] \\
&= (E_t^P[\mathrm{QV}^c_{t,t+\tau}] - E_t^Q[\mathrm{QV}^c_{t,t+\tau}]) + (E_t^P[\mathrm{QV}^j_{t,t+\tau}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}]) \\
&= \mathrm{IVRP}^c_{t,t+\tau} + \mathrm{IVRP}^j_{t,t+\tau}.
\end{aligned}$$

Unreported results show that the continuous part $\mathrm{IVRP}^c_{t,t+\tau}$ contributes more than $\mathrm{IVRP}^j_{t,t+\tau}$ to IVRP, which is not surprising given that jumps are rare events. However, negative jumps may have an important role in determining level and dynamic of IVRP. As most investors are "long in the market" and the leverage effect is very pronounced, negative jump prices are perceived by investors as unfavorable events and thus can carry significant risk premiums. The contribution of negative jumps to the IVRP can be measured by

$$\mathrm{IVRP}(k)^j_{t,t+\tau} = E_t^P[\mathrm{QV}^j_{t,t+\tau}\,1_{\{J<k\}}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}\,1_{\{J<k\}}]$$

where $1_{\{J<k\}}$ is the indicator function of the event $J < k$. We set $k = -1\%$, i.e., we study the contribution of jumps below -1%.[18] Similar values for the threshold k give very similar results.

Table 6 reports summary statistics about the contribution of jumps below 1% to the IVRP. Although such jumps are quite rare events, their contribution to the IVRP is substantial, especially for the 2-month IVRP. Figure 4 shows the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ over time. Similarly to the IVRP, the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is generally downward sloping during quite times. However, in contrast to IVRP, during market crases the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ becomes suddenly upward sloping, reflecting the immediate impact of such negative returns on current variance.

Recently, Bollerslev and Todorov (2011) study the contribution of negative jump to the IVRP using a model-free approach, but for a single, short time horizon. Our results are model-based, but are informative about the dynamic of the term structure of the IVRP.

[18]From January 1996 to September 2010, daily S&P 500 returns are on average 3 times a month below -1%, having a standard deviation of 1.4%.

5.3. Impact of Continuous and Jump Risk on the Term Structure of VS Rates

The contribution of the continuous and jump parts of the quadratic variation to VS rates is given by

$$\frac{E_t^Q[\mathrm{QV}^c_{t,t+\tau}]}{E_t^Q[\mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau}]}, \qquad \frac{E_t^Q[\mathrm{QV}^j_{t,t+\tau}]}{E_t^Q[\mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau}]}$$

respectively. Using Model (6), the ratios above can be calculated analytically for any maturity τ, and this allows to study the impact on the term structure of VS rates of diffusive and jump risk. Not surprisingly $\mathrm{QV}^c_{t,t+\tau}$ accounts for the largest fraction of VS rates, between 80% and 90%, and the longer the maturity the higher the contribution. The contribution of $\mathrm{QV}^j_{t,t+\tau}$ starts to increase in 2004 and reaches historically high values in 2006–7, almost 20%, during historically low volatility period since 1996.

6. Expectation Hypothesis

An extensive finance literature has investigated whether term structures provide valuable information for predicting future asset prices. For example, several studies investigate whether term structures of interest rates, exchange rates or option implied volatilities can predict future bond yields, spot exchange rates or underlying asset volatilities. The standard approach is to regress the variable of interest y_t (e.g., bond yield at time t) on a constant and its predictor x_t (e.g., forward rate at time $t-1$ with maturity t). In the regression $y_t = \alpha + \beta\, x_t + \mathrm{error}_t$, the hypothesis of interest is whether the predictor is unbiased, i.e., $\alpha = 0$, and/or efficient, i.e., $\beta = 1$. In this case, the so-called Expectation Hypothesis (EH) holds.

In our setting a natural question is whether the variance swap rate provides unbiased and/or efficient prediction of future realized variance. To simplify the notation, let y_t denote the annualized realized variance between t and $t+\tau$, and x_t the variance swap rate at time t with maturity $t+\tau$. At time t, the goal is to forecast the random variable y_t with the known predictor x_t.

The traditional empirical approach would be to collect data for y_t and x_t at different times t and then run the regression $y_t = \alpha + \beta\, x_t + \mathrm{error}_t$. The main advantage of this approach is that, of course, the results are data driven. However, various issues, such as incorrectly specifying the regression model, potential outlying observations, limited sample size or lack of statistical power, could contaminate the regression results.

A theoretical, model-based analysis of the EH overcomes such issues. Using Model (5)–(6), we can calculate the theoretical values of α and β, which can be interpreted as the regression outcome when an infinite amount of clean data points are available. It is well-known that such population values are given by $\alpha = E^P[y_t] - \beta E^P[x_t]$, and $\beta = \mathrm{Cov}^P[x_t, y_t]/\mathrm{Var}^P[x_t]$. Additionally, we can calculate a statistic similar to a theoretical R-squared defined as $R^2 = \mathrm{Var}^P[\beta\, x_t]/\mathrm{Var}^P[y_t]$, which measures the fraction of variance of y_t explained by the variance of $\beta\, x_t$.

Intuitively, variance and jump risk premiums imply that VS rates will not predict future realized variances exactly. Our goal is to quantify such inaccuracies, understand the sources, and study the impact of the time horizon on such predictions. To illustrate these issues, we use the Heston model. Under this model, the slope, intercept and R^2 have remarkably simple expressions:

$$
\begin{aligned}
\beta &= \frac{\mathrm{Cov}^P[\mathrm{VS}_{t,t+\tau}, \frac{1}{\tau}\int_t^{t+\tau} v_s\, ds]}{\mathrm{Var}^P[\mathrm{VS}_{t,t+\tau}]} = \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)} \\
\alpha &= E^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] - \beta E^P[\mathrm{VS}_{t,t+\tau}] = \theta_v^P - \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)}((1-\phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)\theta_v^P) \\
R^2 &= \frac{\beta^2 \mathrm{Var}^P[\mathrm{VS}_{t,t+\tau}]}{\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds]} = \phi_v^P(\tau)^2 / \left(\frac{2\,(e^{-k_v^P \tau} - 1 + k_v^P \tau)}{\tau^2\,(k_v^P)^2}\right)
\end{aligned}
$$

where $\theta_v^P = \theta_v^Q\, k_v^Q / k_v^P$, $\phi_v^P(\tau)$ is defined in Section 2, and $\phi_v^Q(\tau)$ is the same function as $\phi_v^P(\tau)$ with parameter k_v^Q. Appendix B derives the expressions above. As β, α and R^2 obviously depend on the time horizon, τ, below we use the notation $\beta(\tau)$, $\alpha(\tau)$ and $R^2(\tau)$.

Plugging (unreported) parameter estimates for the Heston model into $\beta(\tau)$, $\alpha(\tau)$ and $R^2(\tau)$, we can discuss the impact of the time horizon on such quantities, plotted in Figure 5.[19] When $\tau \to 0$, $\beta(\tau) \to 1$ and $\alpha(\tau) \to 0$. Thus, in the Heston model, for very short maturities VS rates are efficient and unbiased predictor of future realized variance. This result is due to the annualized realized variance, $\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds$, converging to the spot variance v_t when $\tau \to 0$. When $\tau \to +\infty$, $\beta(\tau) \to k_v^Q/k_v^P$ and $\alpha(\tau) \to 0$. Hence, for very long maturities VS rates are inefficient but unbiased predictor of future realized variance. As $k_v^Q = k_v^P + \gamma_2\sigma_v$ and γ_2 is estimated to be negative, $k_v^Q/k_v^P < 1$. This suggests that VS rates tend to overestimate future realized variance. Thus, predictions based on VS rates need to be "discounted" by a slope factor, $\beta(\tau)$, which is less than 1. This result is also consistent with a negative unconditional variance risk premium, i.e., $\theta_v^P - \theta_v^Q < 0$. As a function of τ, the theoretical $R^2(\tau)$ behaves similarly to $\beta(\tau)$.

[19]Given the analytical expressions for $\beta(\tau)$, $\alpha(\tau)$, and $R^2(\tau)$, we can go beyond the 2- to 24-month maturity range of VS rates available in our sample.

The decay to zero when the time horizon increases implies that the variance of the regressors is approaching zero faster than $\text{Var}^P[y_t]$.

For the general Model (5)–(6), Appendix B provides analytical expressions for

$$\begin{aligned}\beta(\tau) &= \frac{\text{Cov}^P[\text{VS}_{t,t+\tau}, (\text{QV}^c_{t,t+\tau} + \text{QV}^j_{t,t+\tau})]}{\text{Var}^P[\text{VS}_{t,t+\tau}]} \\ \alpha(\tau) &= E^P[\text{QV}^c_{t,t+\tau} + \text{QV}^j_{t,t+\tau}] - \beta(\tau)E^P[\text{VS}_{t,t+\tau}] \\ R^2(\tau) &= \frac{\beta(\tau)^2\text{Var}^P[\text{VS}_{t,t+\tau}]}{\text{Var}^P[\text{QV}^c_{t,t+\tau} + \text{QV}^j_{t,t+\tau}]}.\end{aligned}$$

Expressions for nested models can be obtained by imposing the corresponding parameter restrictions. Unreported results show that under the two-factor model with no jump component, $\beta(\tau)$, $\alpha(\tau)$, and $R^2(\tau)$ behave qualitatively the same as in the Heston model.

Allowing for the jump component changes the limit behavior of these functions completely. For example, when $\tau \to 0$, the theoretical $R^2(\tau) \to 0$, rather than 1.[20] Also, when $\tau \to 0$, $\beta(\tau)$ goes to $(1 + \lambda_1 E^P[J^2])/(1 + \lambda_1 E^Q[J^2])$, which is different from 1 unless the random jump size J has the same second moment under P and Q. As this limit does not depend on λ_0, for very short maturities VS rates are still efficient predictor of realized variance when a constant jump-intensity component is present (i.e., $\lambda_0 \neq 0$), but not when a stochastic jump-intensity component (i.e., $\lambda_1 \neq 0$) driven by the spot variance is present. As shown in Appendix B, when $\tau \to 0$ or ∞, the theoretical intercept $\alpha(\tau)$ no longer tends to zero.

The analysis above shows that the Expectation Hypothesis does not hold theoretically, i.e., VS rates are biased and inefficient predictors of future realized variances under Model (5)–(6). However, such bias and inefficiency might be modest in practice, suggesting that VS rates still provide reliable predictions of future realized variances. To quantify these aspects, Figure 5 shows $\beta(\tau)$, $\alpha(\tau)$, based on our parameter estimates. The general model implies quite different values for $\beta(\tau)$, $\alpha(\tau)$, and $R^2(\tau)$ than the Heston model. For the range of VS rate maturities in our sample (up to 2 years), $\beta(\tau)$ decreases faster than in Heston model. Hence, predictions of future realized variances based on VS rates appear to deteriorate more quickly than what predicted by the Heston model. In other words, when τ increases, VS rates should be "discounted" more heavily than what suggested by the Heston model to obtain accurate prediction of future realized variance. When $\tau \approx 6$ years, $\beta(\tau)$ in Heston model has already

[20]This phenomenon is due to the variance of the non-annualized $\text{QV}^j_{t,t+\tau}$, i.e., $\text{Var}^P[\tau\,\text{QV}^j_{t,t+\tau}]$, which goes to zero at speed τ only. Thus, the variance of the annualized $\text{QV}^j_{t,t+\tau}$, i.e., $\tau^{-2}\text{Var}^P[\tau\,\text{QV}^j_{t,t+\tau}]$, goes to infinity when $\tau \to 0$, which implies that $R^2(\tau) \to 0$.

reached its asymptotic level (k_v^Q/k_v^P in that model) while in the stochastic jump-intensity two-factor model it has not.

Although $\alpha(\tau)$ has different shapes under the two models, its magnitude, plotted in percentage, is fairly small when compared to percentage VS rates which are around 4 in variance percentage units. Theoretical $R^2(\tau)$ turns out to be monotonically decreasing for a wide range of relevant time to maturities. Although $R^2(\tau) \to 0$ when $\tau \to 0$, given our parameter estimates, this phenomenon becomes important only for very short time to maturities and does not appear to be practically relevant. The level of $R^2(\tau)$ is quite different in the Heston and the general model. For example when $\tau = 2$ years, the $R^2(\tau)$ in the Heston model is only around 35%, while it is almost 70% in the general model. This difference suggests that the stochastic long run mean and jump component have empirically large impact on the Expectation Hypothesis. It also suggests that the variability in VS rates, as measured by $\beta(\tau)\mathrm{VS}_{t,t+\tau}$, reflect more the variability in realized variances.

Using the regression notation, the $\mathrm{IVRP}_{t,t+\tau} = E_t^P[y_t] - x_t$. If $\mathrm{IVRP}_{t,t+\tau}$ was zero, then $\alpha(\tau) = 0$ and $\beta(\tau) = 1$. If $\mathrm{IVRP}_{t,t+\tau}$ was constant over time, then $\alpha(\tau) \neq 0$ and $\beta(\tau) = 1$. Since the analysis above implies that $\beta(\tau) < 1$, theoretically the IVRP is time-varying.

For completeness, we also run the regression $y_t = \alpha + \beta\, x_t + \mathrm{error}_t$ using actual daily data, i.e., VS rates and realized variances. We consider the time horizons of 2, 3, 6 and 12 months to maturity. Obviously, the longer the time horizon, the larger the time overlap in the daily regression variables. Given the strong persistence in these variables, the empirical regression outcome should be interpreted cautiously. The slope estimates are 0.67, 0.57, 0.41 and 0.28, respectively, and are all well below 1, in agreement with the theoretical analysis of the EH. These values are also below the corresponding theoretical values which be due to several reasons, such as limited sample size or other variables missing in the regression model.[21] Estimates of $\alpha(\tau)$, in percentage, are 0.28, 0.86, 1.81 and 2.66, respectively, which are larger than corresponding theoretical values. This might be a reflection of the relatively low estimated values for $\beta(\tau)$.

7. Shorting Variance Swaps

In contrast to stock returns, volatility is partially predictable because it follows a persistent and mean reverting process. For example, if today's volatility is high, it is likely that it will

[21] Since VS rates are actual market data, it is unlikely that the error-in-variable issue might explain this phenomenon.

remain high in the near future, but eventually will revert to lower levels. In that case, a sensible investment might be to go short in a long-term variance swap contract or "sell volatility," speculating that long-term future volatility will be lower than current variance swap rate. This strategy can be seen as a statistical arbitrage applied to variance swaps. Buyer and seller of variance swaps are, of course, well aware of these phenomena and incorporate them in their pricing of VS contracts. A challenge for both sides of the contract is to accurately estimate the speed of mean reversion of the volatility, a notoriously difficult parameter, and its long run mean level. Continuing the example above, if an investor estimates a relatively low speed of mean reversion, it may be no longer optimal to "sell volatility," as it will not mean revert quickly enough to lower levels. Accurately estimating future volatility of volatility is also obviously important to assess return profitability.

We consider a simple but robust trading strategy with VS. The trading strategy is robust in the sense that Model (5)–(6) and corresponding estimates are used only to decide whether or not to invest in VS, i.e., to extract a trading signal. In contrast, for example, optimal investments to maximize some expected utility are certainly more sophisticated, but optimal portfolio weights tend to be rather sensitive to model specifications and estimations; see Egloff, Leippold, and Wu (2010) and Filipović, Gourier, and Mancini (2011).

Since realized variances are lower than VS rates on average, this suggests that shorting VS contracts each day would generate a positive return. This simple trading strategy can be refined as follows. At each day t, we compute the expected profit from shorting a VS contract, i.e., $\mathrm{VS}_{t,t+\tau} - E_t^P[\mathrm{QV}_{t,t+\tau}]$. Note that if the parametric model fits exactly VS rates, such a profit is minus the IVRP, i.e., $-\mathrm{IVRP}_{t,t+\tau} = E_t^Q[\mathrm{QV}_{t,t+\tau}] - E_t^P[\mathrm{QV}_{t,t+\tau}]$. Then, we short the VS contract only when the expected profit is large enough and precisely larger than, say, n times its standard deviation. When $n = 0$, we short the VS contract as soon as the expected profit is positive. When $n > 0$, we short the contract less often. The notional amount of the VS is such that for each unit increase of the payoff we receive \$1. When at day t the VS contract is shorted, we compute the actual return from the investment by comparing the VS rate and ex-post realized variance, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$. Since this is a buy and hold strategy (conditional on a model-based signal), transaction costs are unlikely to affect the results and will not be considered. If at day t the VS is not shorted, the return from t to $t + \tau$ is obviously zero. We repeat this procedure for each day t in our sample. Apart from the model estimates obtained using in-sample data, the trading strategy does not use future information. During the out-sample period, the strategy truly uses only information up to day t.

To assess the economic magnitude of the returns from shorting VS, we consider the following trading strategy based on the S&P 500 index. If at time t the VS contract with maturity $t+\tau$ is shorted, we invest \$1 in the S&P 500 index and liquidate the position at time $t+\tau$. Thus, the investment horizon is the same as the one for the VS strategy. The actual return is easily computed using S&P 500 index prices. This procedure is also repeated for each day t in our sample.

The two trading strategies, based on variance swaps and the S&P 500 index, require different initial capitals and thus generate dollar-returns of different magnitudes. However, their Sharpe ratios can be meaningfully compared. Table 7 reports such Sharpe ratios. Shorting VS appears to be significantly more profitable than investing in the S&P 500 index, over the same time horizons. This suggests that VS contracts offer economically important investment opportunities. It also suggests that investors are ready to pay high "insurance premiums" to obtain protection against volatility increases.

When the threshold n increases, the VS is shorted less often[22] and in particular only when the expected payoff is significantly larger than zero, in terms of standard deviations. This assessment relies on Model (5)–(6). As shown in Table 7, Sharpe ratios from investing in VS are nearly uniformly increasing in the threshold n. The main exception is from shorting the VS contract with two months to maturity in the out-sample period, as such positions suffered large losses due the Lehman's collapse. Overall, Model (5)–(6) seems to provide rather valuable information to generate the trading signal.

8. Robustness Checks

The previous change of measure from P to Q for the jump component implies that the mean jump size changes, but not the jump intensity. Now we let the jump intensity be $\lambda_t^P = \lambda_0^P + \lambda_1^P v_t$ under P and $\lambda_t^Q = \lambda_0^Q + \lambda_1^Q v_t$ under Q. The drift under P of the stock price process becomes

$$\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2 \rho v_t + g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t)$$

and jump risk premiums become

$$\mathrm{JRP}_t \quad = \quad g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t)$$

[22]For example, the 2-month VS is shorted 23%, 9%, 3% of the times, when $n = 1, 2, 3$, respectively. These figures increase to 69%, 32% and 13% for the 24-month VS contract. For the remaining VS contracts, the corresponding figures are somewhere in between.

$$\text{IVRP}^j_{t,t+\tau} \quad = \quad E^P[J^2](\lambda_0^P + \lambda_1^P E_t^P[\text{QV}^c_{t,t+\tau}]) - E^Q[J^2](\lambda_0^Q + \lambda_1^Q E_t^Q[\text{QV}^c_{t,t+\tau}])$$

Estimation results of this more general imply very similar dynamic for spot variance, stochastic long run mean, corresponding instantaneous, jump and diffusive risk premiums, and integrated risk premiums due to the continuous part of the quadratic variation. However, the estimated overall risk neutral jump-intensity, λ_t^Q, turns out to be smaller than objective jump-intensity, λ_t^P; Pan (2002) reports the same finding using her stochastic volatility model. These estimates would imply positive jump-timing risk premium which would induce positive IVRP^j. This finding confirms the limited ability of estimating very flexible change of measures.

9. Conclusions

In this paper we first perform a model-free analysis of VS rates quoted by a major broker-dealer in New York City for five different time to maturities and for approximately 14 years period from January 4, 1996 to September 2, 2010. The implications of this model-free analysis is then modeled parametrically via two-factor stochastic volatility and stochastic jump-intensity model since this model captures the observed features of the data with a parsimonious way. Model-based analysis of VS rates shows that the term structure of variance risk premium is negative and generally downward sloping, while the term structure of variance risk premium due to negative jumps is negative, downward sloping in quite times and upward sloping during market crashes. Theoretically, moreover Expectation Hypothesis does not hold but bias and inefficiency are modest for short/medium time to maturities. Our model-consistent investment strategy on VS's shows that the estimation results can be used to generate better investment performance than long-only S&P 500 index investments for the same time period.

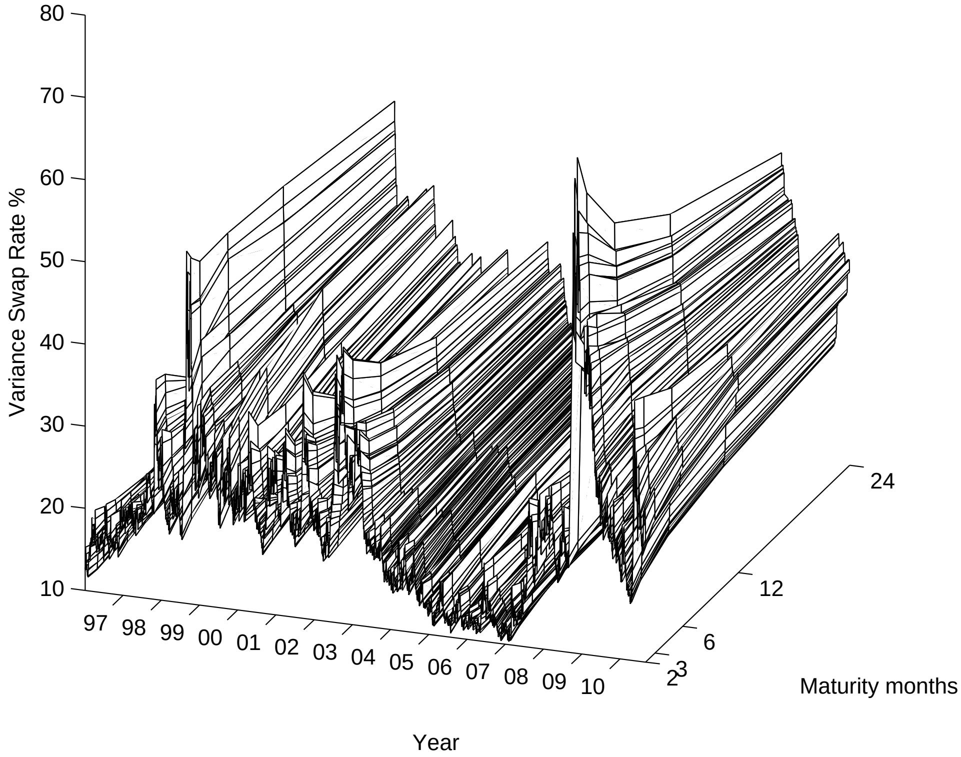

Figure 1. Term structure of variance swap rates. Values are in volatility percentage units with 2-, 3-, 6-, 12-, and 24-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each maturity.

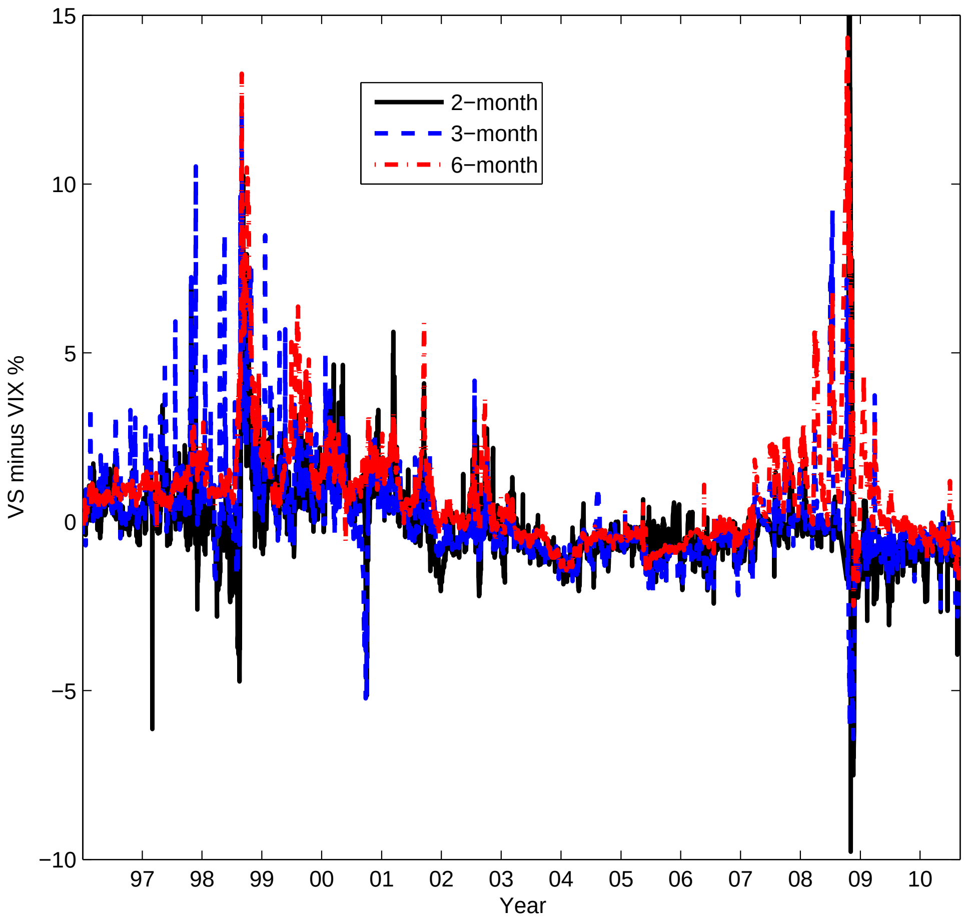

Figure 2. Term structure of model-free jump component in variance swap rates. VS rates minus calculated VIX-type indices for 2-, 3-, and 6-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each maturity. The difference is in volatility percentage units, i.e., $(\mathrm{VS}_{t,t+\tau}^{1/2} - \mathrm{VIX}_{t,t+\tau}^{1/2}) \times 100$.

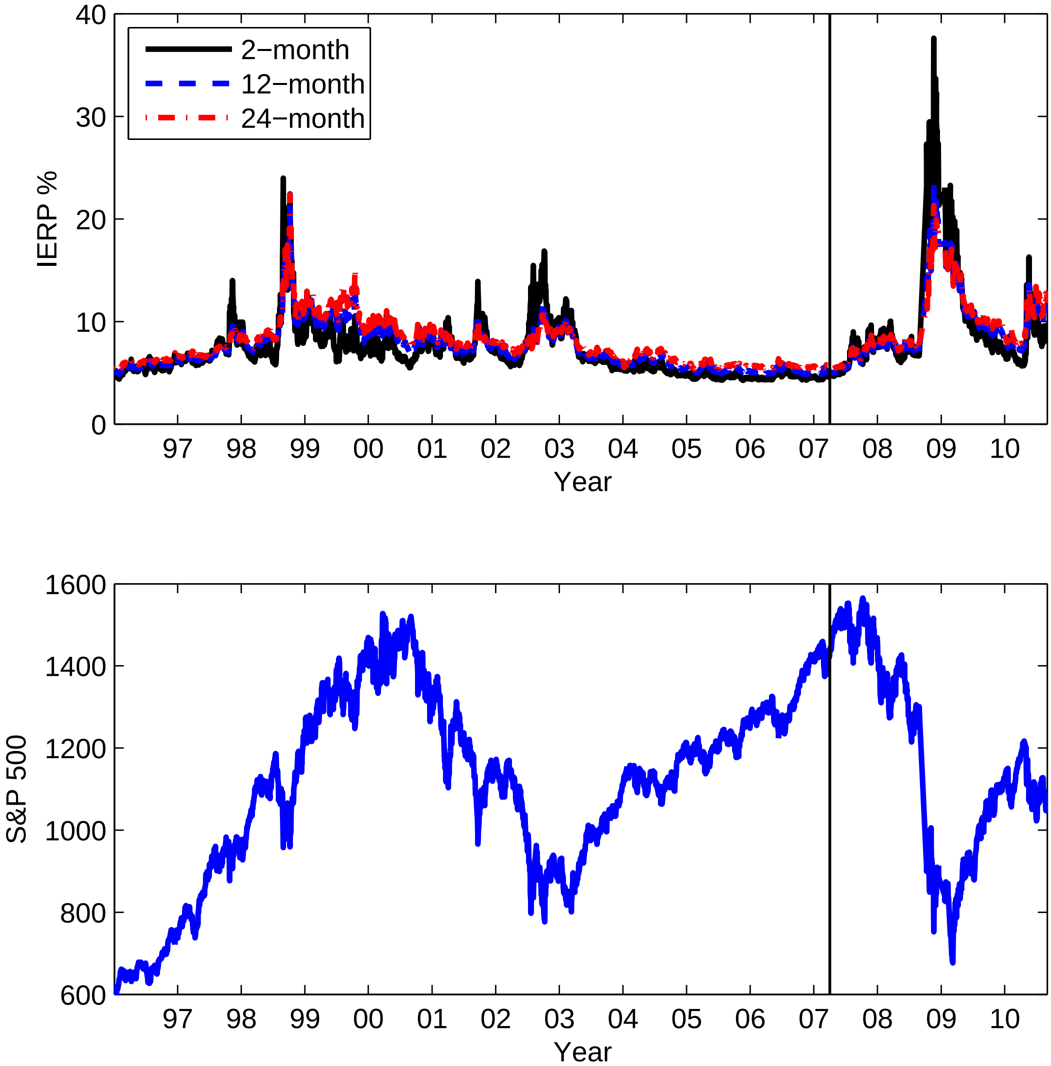

Figure 3. Term structure of integrated equity risk premiums and S&P 500 index. Upper graph: integrated equity risk premiums. Values are annualized and in percentage. Lower graph: S&P 500 index. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

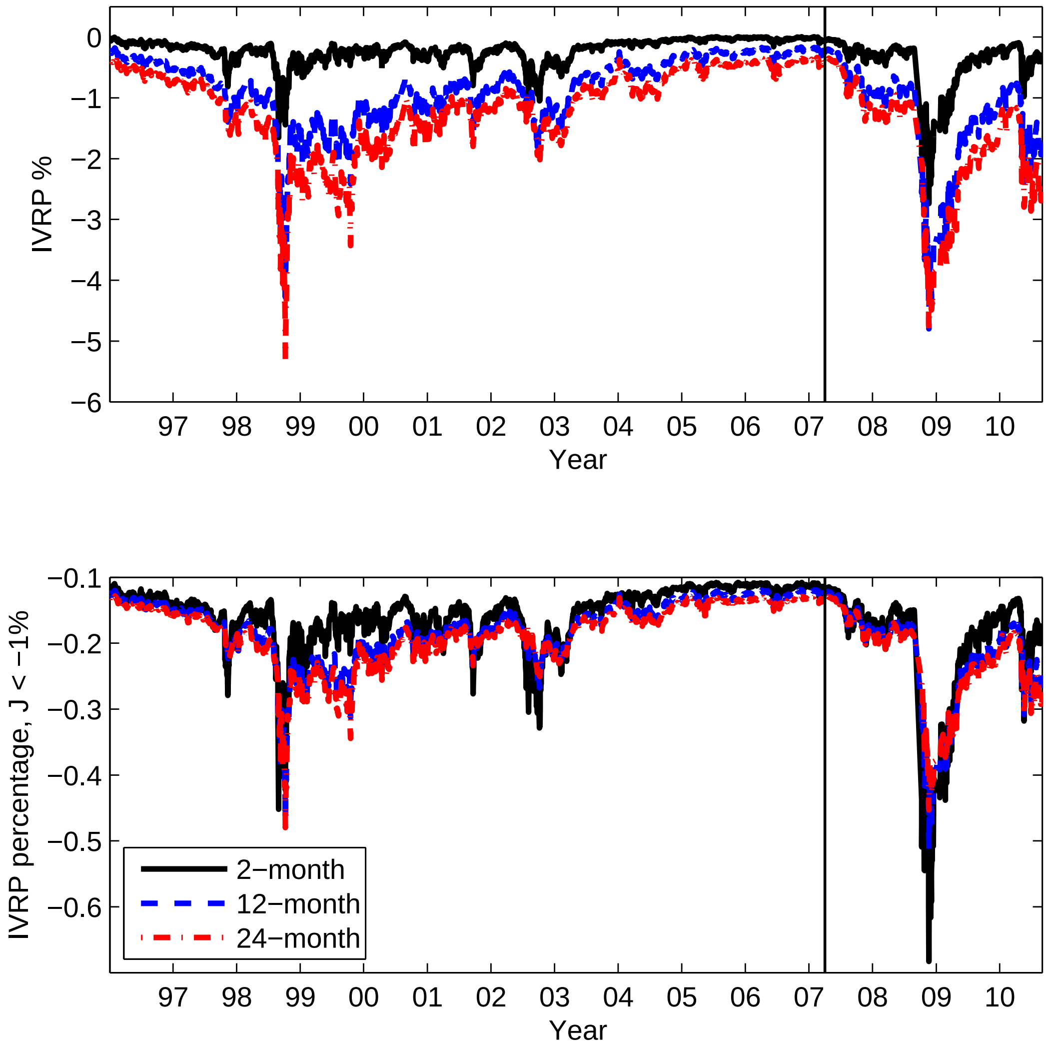

Figure 4. Term structure of integrated variance risk premiums. Upper graph: integrated variance risk premiums. Lower graph: integrated variance risk premium due to jump price below -1%. Values are in variance percentage units. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

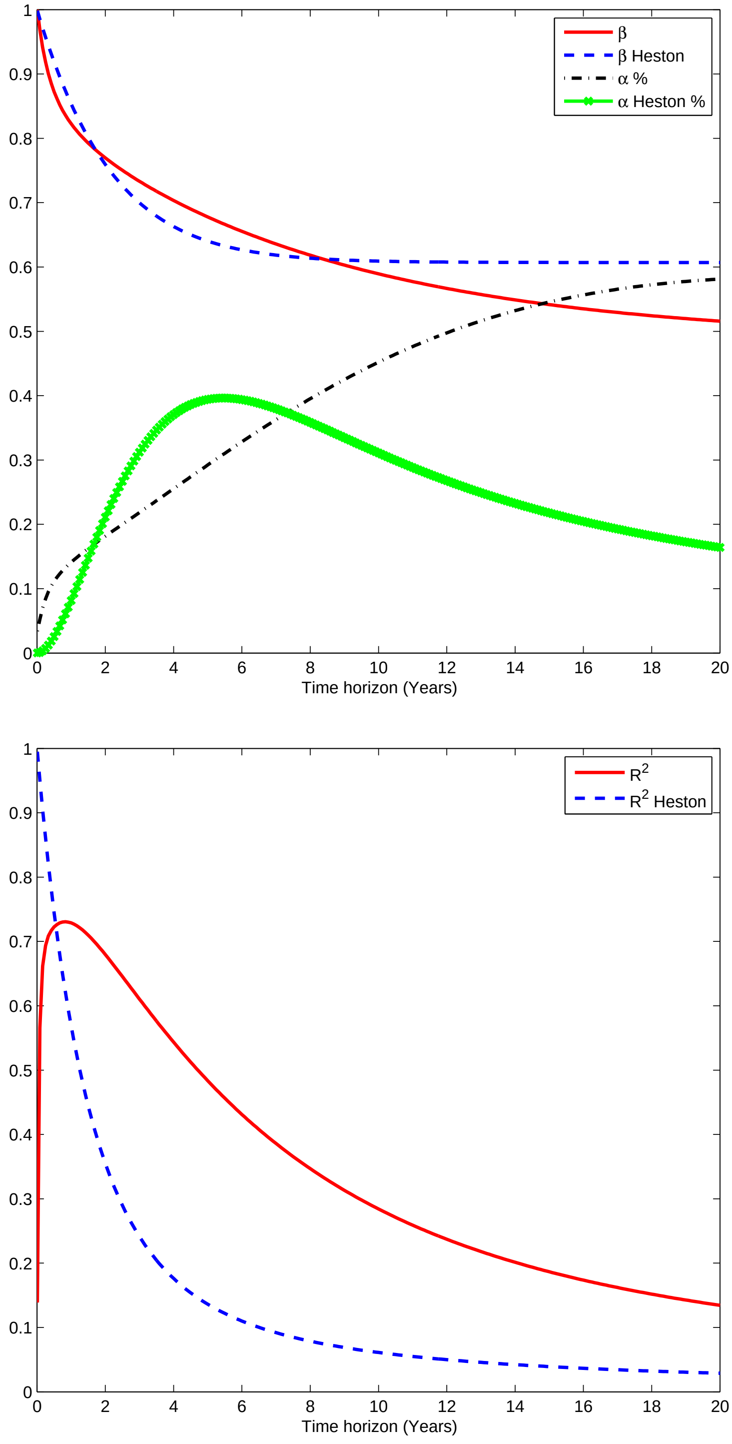

Figure 5. Theoretical intercept, slope and R^2 for the Expectation Hypothesis. Upper graph: theoretical $\alpha(\tau)$, $\beta(\tau)$ for the Model (5)–(6) and the Heston model. Lower graph: theoretical $R^2(\tau)$ for the two models. Time horizon, τ, is expressed in years and ranges between 0.1 months and 20 years.

Maturity	Mean	Std	Skew	Kurt	AC1	Q_{22}	ADF
Panel A: Variance swap rates							
2	22.14	8.18	1.53	7.08	0.982	62,908.97	−3.79
3	22.32	7.81	1.32	6.05	0.988	66,449.22	−3.52
6	22.87	7.40	1.10	4.97	0.992	69,499.72	−3.30
12	23.44	6.88	0.80	3.77	0.994	71,644.69	−2.82
24	23.93	6.48	0.57	2.92	0.995	72,878.68	−2.47
Panel B: Calculated VIX-type indices							
2	20.68	6.14	0.71	3.38	0.988	51,361.97	−3.62
3	20.72	5.80	0.63	3.20	0.987	51,654.62	−3.40
6	20.79	5.23	0.47	2.79	0.994	55,105.72	−3.22
Panel C: Realized variance							
2	18.90	12.40	4.31	28.40	0.993	64,293.09	−5.07
3	19.06	12.04	3.80	21.81	0.996	68,851.75	−4.21
6	19.46	11.33	2.93	13.17	0.998	72,697.32	−2.86
12	20.13	10.47	1.97	6.86	0.999	72,239.41	−1.98
24	20.60	8.81	1.09	3.48	0.998	66,759.84	−0.57

Table 1. Panel A: Summary statistics of the variance swap rates on the S&P 500 index at different maturities in months from January 4, 1996 to September 2, 2010 for a total of 3,624 observations for each maturity. The table reports mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), first order autocorrelation (AC1) the Ljung–Box portmanteau test for up to 22nd order autocorrelation (Q_{22}), 10% critical value is 30.81; the augmented Dickey–Fuller test for unit root involving 22 augmentation lags, a constant term and time trend (ADF), 10% critical value is −3.16. Panel B: summary statistics of the two-, three- and six-month VIX-type indices calculated using SPX options and applying the revised CBOE VIX methodology. Panel C: summary statistics of ex-post realized S&P 500 realized variance for various maturities. All variables are in volatility percentage units.

Parameter	True Value	Mean Bias	RMSE
κ_v^P	3.00	0.34	0.67
σ_v	0.25	0.02	0.02
κ_m^P	0.30	0.11	0.25
θ_m^P	0.10	0.03	0.81
σ_m	0.10	0.01	0.01
ρ	-0.80	0.01	0.01
λ_0	4.00	-1.17	1.47
λ_1	10.00	-4.99	5.32
γ_1	-7.00	6.18	8.44
γ_2	-6.00	-0.85	2.25
γ_3	-1.00	0.91	2.17
μ_j^P	-0.01	-0.01	0.02
μ_j^Q	-0.20	-0.04	0.07
σ_j	0.04	0.03	0.04

Table 2. Monte Carlo simulation results. Model (5)–(6) is simulated at an intraday frequency of 15 minutes using an Euler discretization. To obtain a daily sample, one value every 30 values is picked from the simulated trajectory. VS rates are calculated for 3- and 12-month to maturity using the simulated values. Each sample consists of N = 10,000 data points of the underlying stock price and VS rates. Model (5)–(6) is estimated using each simulated path and the method described in Section 3.

Parameter	Estimate	Std.Err.
κ_v^P	4.803	0.353
σ_v	0.419	0.009
κ_m^P	0.234	0.086
θ_m^P	0.043	0.016
σ_m	0.141	0.002
ρ	-0.713	0.010
λ_0	3.669	0.621
λ_1	44.770	17.227
γ_1	-2.545	4.206
γ_2	-2.244	0.851
γ_3	-0.673	0.610
μ_j^P	0.010	0.008
μ_j^Q	-0.001	0.009
σ_j	0.038	0.003
σ_{e_1}	0.004	0.000
σ_{e_2}	0.002	0.000
σ_{e_3}	0.003	0.000
ρ_e	-0.088	0.006
Log-likelihood	74,381.8	

Table 3. Estimation results for the Model (5)–(6) using the procedure described in Section 3. Variance swap rates with 2-, 3-, 6-, 12-, and 24-month to maturity and S&P 500 returns range from January 4, 1996 to April 3, 2007. Variance swap rates with 3- and 12-month to maturity are assumed to be observed without errors. Variance swap rates with 2-, 6-, and 24-month to maturity are assumed to be observed with errors whose standard deviations are σ_{e_1}, σ_{e_2} and σ_{e_3}, and correlation ρ_e.

	Mean	RMSE	Mean	RMSE
	In-sample			
	Heston		SJSV	
$\widehat{\mathrm{VS}}_{\mathrm{2m}} - \mathrm{VS}_{\mathrm{2m}}$	−0.081	0.851	−0.183	0.743
$\widehat{\mathrm{VS}}_{\mathrm{6m}} - \mathrm{VS}_{\mathrm{6m}}$	0.002	1.119	0.064	0.394
$\widehat{\mathrm{VS}}_{\mathrm{24m}} - \mathrm{VS}_{\mathrm{24m}}$	1.001	2.950	0.143	0.560
	Out-sample			
	Heston		SJSV	
$\widehat{\mathrm{VS}}_{\mathrm{2m}} - \mathrm{VS}_{\mathrm{2m}}$	0.259	1.420	0.218	1.005
$\widehat{\mathrm{VS}}_{\mathrm{6m}} - \mathrm{VS}_{\mathrm{6m}}$	−0.258	1.403	−0.249	0.475
$\widehat{\mathrm{VS}}_{\mathrm{24m}} - \mathrm{VS}_{\mathrm{24m}}$	0.469	3.074	0.129	0.551

Table 4. Variance swap pricing errors. The pricing error is defined as the model-based VS rate minus observed VS rate, in volatility percentage units, i.e., $(E_t^Q[\mathrm{QV}_{t,t+\tau}]^{1/2} - \mathrm{VS}_{t,t+\tau}^{1/2}) \times 100$. The table reports mean, root mean square error of pricing errors for VS rate with 2-, 6- and 24-month to maturity under the Heston model and Model (5)–(6). In-sample period, used to estimate the models, ranges from January 4, 1996 to April 2, 2007. Out-sample period ranges from April 3, 2007 to September 2, 2010.

	In-sample		Out-sample	
	Mean	Std	Mean	Std
DRP	1.27	1.18	2.64	2.82
JRP	5.53	1.58	7.36	3.76
VRP	−3.40	3.16	−7.06	7.54
LRMRP	−0.44	0.30	−0.68	0.38

Table 5. Instantaneous risk premiums. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t$; Jump risk premium $\mathrm{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0 + \lambda_1 v_t)$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\mathrm{LRMRP}_t = \gamma_3\sigma_m m_t$. Risk premiums are based on Model (5)–(6). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-sample period ranges from April 3, 2007 to September 2, 2010. Entries are in percentage.

	In-sample		Out-sample	
Maturity	Mean	Std	Mean	Std
	Equity			
2	6.90	2.37	9.68	5.30
6	7.07	2.13	9.43	4.09
12	7.31	2.09	9.42	3.48
24	7.77	2.10	9.74	3.17
	Jump contribution			
2	5.61	1.37	7.22	3.07
6	5.76	1.25	7.14	2.41
12	5.97	1.25	7.24	2.10
24	6.39	1.31	7.62	2.00
	Variance			
2	−0.21	0.19	−0.43	0.44
6	−0.53	0.39	−0.96	0.75
12	−0.80	0.53	−1.33	0.86
24	−1.11	0.66	−1.71	0.94
	$J < -1\%$ contribution			
2	−0.15	0.04	−0.20	0.09
6	−0.16	0.04	−0.21	0.08
12	−0.17	0.05	−0.22	0.07
24	−0.18	0.05	−0.23	0.07

Table 6. Term structure of integrated equity and variance risk premiums. Integrated equity risk premium $\text{IERP}_t = E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau$. Integrated variance risk premium $\text{IVRP}_t = E_t^P[\text{QV}_{t,t+\tau}] - E_t^Q[\text{QV}_{t,t+\tau}]$. Risk premiums are based on Model (5)–(6). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-sample period ranges from April 3, 2007 to September 2, 2010. Variance risk premiums are in variance units. All entries are in percentage.

	In-sample							
	Short Variance Swap				Long S&P 500			
Horizon	2	3	6	12	2	3	6	12
Threshold								
Always	0.56	0.59	0.67	0.84	0.22	0.26	0.32	0.27
0	0.55	0.59	0.67	0.84	0.17	0.26	0.32	0.27
1	0.60	0.71	1.15	0.94	0.20	0.24	0.62	0.08
2	0.78	1.35	1.91	1.75	0.61	0.85	1.22	0.27
3	0.98	1.60	2.14	2.71	0.79	1.52	1.72	1.36
	Out-sample							
	Short Variance Swap				Long S&P 500			
Always	0.20	0.15	0.08	0.02	0.05	−0.03	−0.05	−0.18
0	0.48	0.15	0.08	0.02	0.17	−0.03	−0.05	−0.18
1	0.45	0.19	0.10	0.05	0.44	0.04	0.01	−0.12
2	0.20	1.13	1.15	1.46	0.23	0.54	0.99	0.85
3	−0.10	0.91	1.72	2.79	−0.11	0.29	1.18	1.75

Table 7. Sharpe ratios from short positions in variance swaps and long positions the S&P 500 index. For each day t in the sample, the expected profit from a short position in a VS contract is computed, i.e., $\mathrm{VS}_{t,t+\tau} - E_t^P[\mathrm{QV}_{t,t+\tau}]$. If the expected profit is n times larger than its standard deviation, then the VS contract is shorted. Otherwise no position is taken at day t. The column "Threshold" reports the number of standard deviations n. "Always" means the VS contracted is always shorted. At time $t+\tau$, the actual profit is computed, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$. The notional amount in the VS contract is such that for each unit increase of the payoff, the short side receives \$1. The investment strategy in the S&P 500 is as follows. If at day t the VS contract with maturity $t+\tau$ is shorted, \$1 is invested in the S&P 500 at time t. The position is held until $t+\tau$ and then liquidated. Sharpe ratios are computed using all the returns from each investment strategy and assuming a risk free rate of 4%. VS contracts with 2-, 3-, 6-, and 12-month to maturities are considered. The row "Horizon" reports the time to maturity. In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-sample period ranges from April 3, 2007 to September 2, 2010.

A. Pricing Kernel

Recall that the market price of risks for the Brownian motions are

$$\Lambda_t = [\gamma_1\sqrt{(1-\rho^2)v_t}, \quad \gamma_2\sqrt{v_t}, \quad \gamma_3\sqrt{m_t}]'.$$

The jump size risk premium is $(g^P - g^Q) = \exp(\mu_j^P + \sigma_j^2/2) - \exp(\mu_j^Q + \sigma_j^2/2)$. Note that in the P-dynamic of the stock price

$$\begin{aligned}
\mu_t - \nu_t^P &= r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t - g^P\lambda_t \\
&= r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t
\end{aligned}$$

the term $g^P\lambda_t$ cancels.

The pricing kernel (or Stochastic Discount Factor) is defined as

$$\pi_t = e^{-rt}\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-rt - \int_0^t \Lambda_s'\, dW_s^P - \frac{1}{2}\int_0^t \Lambda_s'\Lambda_s\, ds\right)\prod_{u=1}^{N_t}\exp\left(a_j + b_j J_u^P\right)$$

where

$$a_j = \frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2}, \qquad b_j = \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}.$$

The process π_t is a valid pricing kernel when deflated bank account and deflated cum-dividend price processes are P-martingales.

When a jump occurs the pricing kernel jumps from $\pi_{t\_}$ to $\pi_t = \pi_{t\_}e^{a_j + b_j J_t^P}$, hence

$$\begin{aligned}
\frac{d\pi_t}{\pi_t} &= -r\,dt - \Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^P) - 1)\,dN_t^P \\
&= -r\,dt - (\gamma_1\sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \gamma_2\sqrt{v_t}\,dW_{2t}^P + \gamma_3\sqrt{m_t}\,dW_{3t}^P) + (\exp(a_j + b_j J_t^P) - 1)\,dN_t^P
\end{aligned}$$

Let $B_t = e^{rt}$ denote the bank account and $B_t^\pi = B_t\pi_t$ the deflated bank account. Applying Itô's formula

$$\begin{aligned}
d(B_t^\pi) &= B_t\,d\pi_t + \pi_t\,dB_t \\
&= B_t^\pi(-r\,dt - \Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^P) - 1)\,dN_t^P) + B_t^\pi r\,dt \\
d(B_t^\pi)/B_t^\pi &= -\Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^P) - 1)\,dN_t^P
\end{aligned}$$

Hence B_t^π is a P-martingale (or has zero drift) when $E^P[\exp(a_j + b_j J_t^P)] = 1$ which is the case as shown in the following calculations:

$$E^P[\exp(a_j + b_j J_t^P)] = \exp(a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2})$$

$$
\begin{aligned}
a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} &= \frac{(\mu_j^P)^2-(\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q-\mu_j^P}{\sigma_j^2}\mu_j^P + \left(\frac{\mu_j^Q-\mu_j^P}{\sigma_j^2}\right)^2\frac{\sigma_j^2}{2} \\
&= \frac{(\mu_j^P)^2-(\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q-\mu_j^P}{\sigma_j^2}\mu_j^P + \left(\frac{(\mu_j^Q)^2+(\mu_j^P)^2-2\mu_j^Q\mu_j^P}{\sigma_j^2}\right)\frac{1}{2} \\
&= \frac{(\mu_j^P)^2-(\mu_j^Q)^2+2\mu_j^Q\mu_j^P-2(\mu_j^P)^2+(\mu_j^Q)^2+(\mu_j^P)^2-2\mu_j^Q\mu_j^P}{2\sigma_j^2} = 0
\end{aligned}
$$

where we use $J^P \sim \mathcal{N}(\mu_j^P, \sigma_j^2)$.

Let $S_{\delta,t} = S_t e^{\delta t}$ denote the cum-dividend stock price, hence

$$
\begin{aligned}
\frac{dS_{\delta,t}}{S_{\delta,t}} &= \frac{dS_t}{S_t} + \delta\, dt \\
&= (r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\, dt + \sqrt{(1-\rho^2)v_t}\, dW_{1t}^P + \rho\sqrt{v_t}\, dW_{2t}^P + (\exp(J_t^P)-1)\, dN_t^P
\end{aligned}
$$

where S_t is the ex-dividend stock price. Let $S_{\delta,t}^\pi$ be the deflated cum-dividend stock price. When a jump occurs both π_t and S_t jump, and S_δ^π jumps from S_{δ,t_-}^π to $S_{\delta,t}^\pi = S_{\delta,t_-}^\pi \exp(a_j + b_j J_t^P + J_t^P)$. Hence at the jump time $dS_{\delta,t}^\pi / S_{\delta,t}^\pi = \exp(a_j + (b_j+1)J_t^P) - 1$.

Applying Itô's formula, with π_t^c and $S_{\delta,t}^c$ denoting the continuous part of π_t and $S_{\delta,t}$, respectively,

$$
\begin{aligned}
d(S_{\delta,t}^\pi) &= S_{\delta,t}\, d\pi_t^c + \pi_t\, dS_{\delta,t}^c + dS_{\delta,t}^c\, d\pi_t^c + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^P) - 1)\, dN_t^P \\
&= S_{\delta,t}\,\pi_t(-r\, dt - \gamma_1\sqrt{(1-\rho^2)v_t}\, dW_{1t}^P - \gamma_2\sqrt{v_t}\, dW_{2t}^P - \gamma_3\sqrt{m_t}\, dW_{3t}^P) \\
&\quad + \pi_t\, S_{\delta,t}((r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\, dt + \sqrt{(1-\rho^2)v_t}\, dW_{1t}^P + \rho\sqrt{v_t}\, dW_{2t}^P) \\
&\quad - S_{\delta,t}\pi_t(\gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t)\, dt + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^P) - 1)\, dN_t^P \\
\frac{d(S_{\delta,t}^\pi)}{(S_{\delta,t}^\pi)} &= \sqrt{(1-\rho^2)v_t}(1-\gamma_1)\, dW_{1t}^P + (\rho-\gamma_2)\sqrt{v_t}\, dW_{2t}^P - \gamma_3\sqrt{m_t}\, dW_{3t}^P \\
&\quad + (\exp(a_j + (b_j+1)J_t^P) - 1)\, dN_t^P - g^Q\lambda_t\, dt
\end{aligned}
$$

Hence $S_{\delta,t}^\pi$ is a P-martingale (or has zero drift) when $E^P[\exp(a_j + (b_j+1)J_t^P) - 1] = g^Q$ which is the case as shown in the following calculations:

$$
\begin{aligned}
E^P[\exp(a_j + (b_j+1)J_t^P) - 1] &= g^Q \\
\exp(a_j + (b_j+1)\mu_j^P + (b_j+1)^2\frac{\sigma_j^2}{2}) - 1 &= \exp(\mu_j^Q + \frac{\sigma_j^2}{2}) - 1 \\
\exp(a_j + b_j\mu_j^P + \mu_j^P + b_j^2\frac{\sigma_j^2}{2} + 2b_j\frac{\sigma_j^2}{2} + \frac{\sigma_j^2}{2}) &= \exp(\mu_j^Q + \frac{\sigma_j^2}{2}) \\
a_j + b_j\mu_j^P + \mu_j^P + b_j^2\frac{\sigma_j^2}{2} + 2b_j\frac{\sigma_j^2}{2} &= \mu_j^Q \\
\mu_j^P + 2b_j\frac{\sigma_j^2}{2} &= \mu_j^Q
\end{aligned}
$$

$$\mu_j^P + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\sigma_j^2 \quad = \quad \mu_j^Q$$

where we use $a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} = 0$ implied by the martingale property of the deflated bank account.

The relation between the pricing kernel π_t and the risk neutral dynamics is derived as usual. Define the density process $\xi_t = \pi_t e^{rt}$. Under usual technical conditions, applying Itô's formula

$$d\xi_t/\xi_t = -\Lambda_t'\, dW_t^P + (\exp(a_j + b_j J_t^P) - 1)\, dN_t^P$$

shows that ξ_t is a P-martingale and hence it uniquely defines an equivalent martingale measure Q. Defining the Q-Brownian motions as

$$\begin{aligned} dW_{1t}^Q &= dW_{1t}^P + \gamma_1\sqrt{(1-\rho^2)v_t}\, dt \\ dW_{2t}^Q &= dW_{2t}^P + \gamma_2\sqrt{v_t}\, dt \\ dW_{3t}^Q &= dW_{3t}^P + \gamma_3\sqrt{m_t}\, dt \end{aligned}$$

gives the risk neutral dynamic of the stock price S, spot variance v, and stochastic long run m in (6).

B. Expectation Hypothesis

B.1. Basic Properties of Heston-type Models

We start by reviewing some basic properties of the Heston-type models. Recall that under P:

$$\begin{aligned} dv_t &= k_v^P(\frac{k_v^Q}{k_v^P}m_t - v_t)dt + \sigma_v\sqrt{v_t}\, dW_{2t} \\ dm_t &= k_m^P(\theta_m^P - m_t)dt + \sigma_m\sqrt{m_t}\, dW_{3t} \end{aligned}$$

and under Q:

$$\begin{aligned} dv_t &= k_v^Q(m_t - v_t)dt + \sigma_v\sqrt{v_t}\, dW_{2t} \\ dm_t &= k_m^Q(\theta_m^Q - m_t)dt + \sigma_m\sqrt{m_t}\, dW_{3t} \end{aligned}$$

Under P, m_t has unconditional Gamma distribution[23] with parameters $a = 2k_m^P\theta_m^P/\sigma_m^2$ and $b = \sigma_m^2/2k_m^P$ which implies

$$E^P[m_t] = ab = \theta_m^P, \qquad \mathrm{Var}^P[m_t] = ab^2 = \frac{\sigma_m^2\theta_m^P}{2k_m^P}$$

[23]The Gamma distribution is defined as $f(m) = m^{a-1}e^{-m/b}/(b^a\Gamma[a])$, where $\Gamma[a]$ is the Gamma function.

and similarly under Q.

Applying Itô's formula to $(e^{k_v^P t} v_t)$ and rearranging terms gives

$$v_T = v_t e^{-k_v^P(T-t)} + k_v^P \int_t^T e^{-k_v^P(T-u)} \frac{k_v^Q}{k_v^P} m_u \, du + \int_t^T e^{-k_v^P(T-u)} \sigma_v \sqrt{v_u} \, dW_{2u}$$

and similarly for m_T, both under P and Q measures. Conditioning on the trajectory of m between t and T, v has a time-varying but deterministic long run mean. Such representations of v_T and m_T are useful for calculating conditional expectations

$$\begin{aligned}
E^P[m_T|m_t] &= m_t \, e^{-k_m^P(T-t)} + \int_t^T e^{-k_m^P(T-u)} k_m^P \theta_m^P \, du \\
&= m_t \, e^{-k_m^P(T-t)} + \theta_m^P (1 - e^{-k_m^P(T-t)}) \\
E^P[v_T|v_t, \{m_u, u \in [t,T]\}] &= v_t \, e^{-k_v^P(T-t)} + \int_t^T e^{-k_v^P(T-u)} k_v^P \frac{k_v^Q}{k_v^P} m_u \, du \\
E^P[v_T|v_t, m_t] &= v_t \, e^{-k_v^P(T-t)} + \int_t^T e^{-k_v^P(T-u)} k_v^P \frac{k_v^Q}{k_v^P} E^P[m_u|m_t] \, du \\
&= v_t \, e^{-k_v^P(T-t)} + \frac{k_v^Q}{k_v^P} m_t \left[k_v^P \frac{e^{-k_m^P(T-t)} - e^{-k_v^P(T-t)}}{k_v^P - k_m^P} \right] \\
&\quad + \frac{k_v^Q}{k_v^P} \theta_m^P \left[1 + \frac{k_m^P e^{-k_v^P(T-t)} - k_v^P e^{-k_m^P(T-t)}}{k_v^P - k_m^P} \right] \\
E_t^P[\frac{1}{\tau} \int_0^\tau v_s \, ds] &= \frac{1}{\tau} \int_0^\tau E^P[v_{t+u}|v_t, m_t] du \\
&= v_t \phi_v^P(\tau) + \frac{k_v^Q}{k_v^P} m_t \phi_m^P(\tau) + \frac{k_v^Q}{k_v^P} \theta_m^P (1 - \phi_v^P(\tau) - \phi_m^P(\tau)) \\
E^P[\frac{1}{\tau} \int_0^\tau v_s \, ds] &= \theta_v^P
\end{aligned}$$

as $E^P[m_t] = \theta_m^P$ and $\theta_v^P = \theta_m^P k_v^Q / k_v^P$. Corresponding expressions hold under Q. Other expectations that will be used below are collected here

$$\begin{aligned}
E^Q[v_t] &= E^Q[m_t] = \theta_m^Q \\
E^P[v_t] &= E^P[E^P[v_t|m_t]] = E^P[\frac{k_v^Q}{k_v^P} m_t] = \frac{k_v^Q}{k_v^P} \theta_m^P = \theta_v^P \\
E^P[m_t] &= \theta_m^P \\
\mathrm{Var}^P[m_t] &= \frac{\sigma_m^2 \theta_m^P}{2k_m^P} \\
\mathrm{Var}^P[v_t] &= E^P[\mathrm{Var}^P[v_t|m_t]] + \mathrm{Var}^P[E^P[v_t|m_t]] = E^P[\frac{\sigma_v^2 \frac{k_v^Q}{k_v^P} m_t}{2k_v^P}] + \mathrm{Var}[\frac{k_v^Q}{k_v^P} m_t] \\
&= \frac{\sigma_v^2 \frac{k_v^Q}{k_v^P} \theta_m^P}{2k_v^P} + \left(\frac{k_v^Q}{k_v^P} \right)^2 \frac{\sigma_m^2 \theta_m^P}{2k_m^P} = \frac{\sigma_v^2 \theta_v^P}{2k_v^P} + \left(\frac{k_v^Q}{k_v^P} \right) \frac{\sigma_m^2 \theta_v^P}{2k_m^P} \\
E^P[v_t m_t] &= E^P[E[v_t m_t|m_t]] = E^P[\frac{k_v^Q}{k_v^P} m_t m_t] = \frac{k_v^Q}{k_v^P} E^P[m_t^2] = \frac{k_v^Q}{k_v^P} (\mathrm{Var}^P[m_t] + (E^P[m_t])^2)
\end{aligned}$$

$$\text{Cov}^P[v_t, m_t] \;=\; E^P[v_t m_t] - E^P[v_t]E^P[m_t] = \frac{k_v^Q}{k_v^P}\text{Var}^P[m] + \frac{k_v^Q}{k_v^P}(\theta_m^P)^2 - \frac{k_v^Q}{k_v^P}\theta_m^P\theta_m^P = \frac{k_v^Q}{k_v^P}\text{Var}^P[m]$$

B.2. Expectation Hypothesis in Heston Model

The Heston model provides a simple framework to easily see derivations of $\alpha(\tau)$, $\beta(\tau)$, and $R^2(\tau)$. We fist discuss execration hypothesis in this model, and then in the stochastic jump-intensity two-factor model.

Recall that in the Heston model under P:

$$dv_t = k_v^P(\theta_v^P - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^P$$

and under Q:

$$dv_t = k_v^Q(\theta_v^Q - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^Q$$

where $k_v^P = k_v^Q - \gamma_2\sigma_v$ and $\theta_v^P = \theta_v^Q\,k_v^Q/k_v^P$.

The following quantities are used in the expectation hypothesis for the Heston model:

$$\begin{aligned}
E^P[y_t] &= E^P[\text{QV}_{t,t+\tau}] = E^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\,ds] = \theta_v^P \\
E^P[x_t] &= E^P[\text{VS}_{t,t+\tau}] = E^P[(1-\phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)v_t] = (1-\phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)\theta_v^P \\
\text{Cov}^P[x_t, y_t] &= E^P[(\text{VS}_{t,t+\tau} - (1-\phi_v^Q(\tau))\theta_v^Q - \phi_v^Q(\tau)\theta_v^P)\,(\text{QV}_{t,t+\tau} - \theta_v^P)] \\
&= E^P[E_t^P[\phi_v^Q(\tau)v_t - \phi_v^Q(\tau)\theta_v^P)\,(\text{QV}_{t,t+\tau} - \theta_v^P)]] \\
&= E^P[\phi_v^Q(\tau)(v_t - \theta_v^P)E_t^P[\text{QV}_{t,t+\tau} - \theta_v^P]] \\
&= E^P[\phi_v^Q(\tau)(v_t - \theta_v^P)[(1-\phi_v^P(\tau))\theta_v^P + \phi_v^P(\tau)v_t - \theta_v^P]] \\
&= E^P[\phi_v^Q(\tau)(v_t - \theta_v^P)[\phi_v^P(\tau)(v_t - \theta_v^P)]] \\
&= \phi_v^Q(\tau)\phi_v^P(\tau)E^P[(v_t - \theta_v^P)^2] \\
&= \phi_v^Q(\tau)\phi_v^P(\tau)\text{Var}^P[v_t] \\
\text{Var}^P[x_t] &= \text{Var}^P[\text{VS}_{t,t+\tau}] = \text{Var}^P[(1-\phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)v_t] \\
&= (\phi_v^Q(\tau))^2\text{Var}^P[v_t]
\end{aligned}$$

Hence

$$\begin{aligned}
\beta(\tau) &= \frac{\text{Cov}^P[x_t, y_t]}{\text{Var}^P[x_t]} = \frac{\phi_v^Q(\tau)\phi_v^P(\tau)\text{Var}^P[v_t]}{(\phi_v^Q(\tau))^2\text{Var}^P[v_t]} = \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)} \\
\alpha(\tau) &= E^P[y_t] - \beta(\tau)E^P[x_t] = \theta_v^P - \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)}[(1-\phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)\theta_v^P]
\end{aligned}$$

where

$$\phi_v^P(\tau) = \frac{1 - e^{-k_v^P \tau}}{k_v^P \tau}, \qquad \phi_v^Q(\tau) = \frac{1 - e^{-k_v^Q \tau}}{k_v^Q \tau}$$

When $\gamma_2 < 0$, as in our estimates, $k_v^P > k_v^Q$ and $\theta_v^P < \theta_v^Q$. For very short maturities

$$\lim_{\tau \to 0} \beta(\tau) = 1$$

which means VS rates are efficient predictor of future realized variance, which is indeed just the spot variance v_t. For very long maturities

$$\lim_{\tau \to \infty} \beta(\tau) = \lim_{\tau \to \infty} \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)} = \frac{k_v^Q}{k_v^P} < 1$$

which means VS rates are inefficient, upward biased predictor of future realized variance. Given estimated parameters (and in particular $\gamma_2 < 0$), $\beta(\tau)$ is monotonically decreasing from 1 to $k^Q/k^P < 1$ when τ goes from 0 to $+\infty$.

For very short maturities

$$\lim_{\tau \to 0} \alpha(\tau) = 0$$

and for very long maturities

$$\lim_{\tau \to \infty} \alpha(\tau) = \theta_v^P - \frac{k^Q}{k^P} \theta_v^Q = 0$$

as $\theta_v^P = \theta_v^Q \, k_v^Q / k_v^P$. Given our estimated parameters, $\alpha(\tau)$ is positive and hump-shaped when $\tau \in (0, +\infty)$, and the hump is still pronounced when $\tau \in (0, 10)$ years. This means that VS rates are biased predictor of future realized variance.

The theoretical R^2 of Expectation hypothesis regression is

$$R^2(\tau) = \frac{\mathrm{Var}^P[\beta(\tau) x_t]}{\mathrm{Var}^P[y_t]} = \frac{\beta(\tau)^2 \mathrm{Var}^P[\mathrm{VS}_{t,t+\tau}]}{\mathrm{Var}^P[\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds]}$$

The numerator of $R^2(\tau)$ can be computed using expressions above. The denominator

$$\mathrm{Var}^P[\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds] = E^P[\Big(\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds\Big)^2] - (\theta_v^P)^2$$

involves $E^P[\Big(\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds\Big)^2]$ which can be calculated in various ways. One possibility is to see $\Big(\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds\Big)^2$ as a double integral on the square $[t, t+\tau] \times [t, t+\tau]$. Then the idea is to split this square in two parts along the 45-degree line, say $u \le s$ and $u \ge s$ for $u, s \in [t, t+\tau]$. As the integrand, $E[v_u \, v_s]$, is the same in these two triangles we have

$$\begin{aligned} E^P[\Big(\frac{1}{\tau} \int_t^{t+\tau} v_s \, ds\Big)^2] &= \frac{1}{\tau^2} E^P[\Big(\int_t^{t+\tau} v_s \, ds\Big) \Big(\int_t^{t+\tau} v_u \, du\Big)] = \frac{1}{\tau^2} E^P[\int_t^{t+\tau} ds \int_t^{t+\tau} du \; (v_u \, v_s)] \\ &= \frac{1}{\tau^2} E^P[\int_t^{t+\tau} ds \int_t^{s} du \; (v_u \, v_s) + \int_t^{t+\tau} ds \int_s^{t+\tau} du \; (v_u \, v_s)] \\ &= \frac{2}{\tau^2} \int_t^{t+\tau} ds \int_t^{s} du \; E^P[v_u \, v_s] \end{aligned} \tag{14}$$

Hence when $u \leq s$

$$
\begin{aligned}
E^P[v_u\, v_s] &= E^P[E^P[v_u\, v_s|v_u]] \\
&= E^P[v_u\, E^P[\left(v_u e^{-k_v^P(s-u)} + k_v^P \int_u^s e^{-k_v^P(s-l)}\theta_v^P\, dl + \int_u^s e^{-k_v^P(s-l)}\sigma_v\sqrt{v_l}\, dW_{2l}^P \right) |v_u]] \\
&= E^P[v_u^2]e^{-k_v^P(s-u)} + k_v^P(\theta_v^P)^2 \int_u^s e^{-k_v^P(s-l)}\, dl
\end{aligned}
$$

where

$$
E^P[v_u^2] = \mathrm{Var}^P[v_u] + (E^P[v_u])^2 = \frac{\sigma_v^2\theta_v^P}{2k_v^P} + (\theta_v^P)^2
$$

The multiple integral in (14) can be quickly calculated with Mathematica

$$
\begin{aligned}
&\frac{2}{\tau^2}\int_t^{t+\tau} ds \int_t^s du\ E^P[v_u\, v_s] \\
&= \frac{2}{\tau^2}\int_t^{t+\tau} ds \int_t^s du \left(E^P[v_u^2]e^{-k_v^P(s-u)} + k_v^P(\theta_v^P)^2 \int_u^s e^{-k_v^P(s-l)}\, dl \right) \\
&= \frac{2}{\tau^2}E^P[v_u^2]\frac{-1+e^{-k_v^P\tau}+k_v^P\tau}{(k_v^P)^2} + \frac{2}{\tau^2}(\theta_v^P)^2\frac{2-2e^{-k_v^P\tau}-2k_v^P\tau+(k_v^P\tau)^2}{2(k_v^P)^2} \\
&= \frac{2}{\tau^2}\frac{(1-e^{-k_v^P\tau}-k_v^P\tau)}{(k_v^P)^2}(-E^P[v_u^2]+(\theta_v^P)^2) + (\theta_v^P)^2 \\
&= -\frac{2}{\tau^2}\frac{(1-e^{-k_v^P\tau}-k_v^P\tau)}{(k_v^P)^2}\,\mathrm{Var}^P[v_u] + (\theta_v^P)^2
\end{aligned}
$$

Hence, denominator of $R^2(\tau)$ is

$$
\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] = \frac{2}{\tau^2}\frac{(e^{-k_v^P\tau}-1+k_v^P\tau)}{(k_v^P)^2}\,\mathrm{Var}^P[v_u].
$$

As expected

$$
\lim_{\tau\to 0}\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] = \mathrm{Var}^P[v_t], \qquad \lim_{\tau\to\infty}\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] = \mathrm{Var}^P[\theta_v^P] = 0.
$$

Collecting the necessary terms

$$
R^2(\tau) = \frac{\left(\frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)}\right)^2(\phi_v^Q(\tau))^2\mathrm{Var}^P[v_t]}{\frac{2}{\tau^2}\frac{(e^{-k_v^P\tau}-1+k_v^P\tau)}{(k_v^P)^2}\,\mathrm{Var}^P[v_u]} = \frac{\left(\phi_v^P(\tau)\right)^2}{\frac{2}{\tau^2}\frac{(e^{-k_v^P\tau}-1+k_v^P\tau)}{(k_v^P)^2}}
$$

which implies that

$$
\lim_{\tau\to 0}R^2(\tau) = 1, \qquad \lim_{\tau\to\infty}R^2(\tau) = 0.
$$

B.3. Expectation Hypothesis in Stochastic Jump-intensity Two-factor Model

Recall that under P:

$$\begin{aligned} dv_t &= k_v^P\left(\frac{k_v^Q}{k_v^P}m_t - v_t\right)dt + \sigma_v\sqrt{v_t}\,dW_{2t} \\ dm_t &= k_m^P(\theta_m^P - m_t)dt + \sigma_m\sqrt{m_t}\,dW_{3t} \end{aligned}$$

and under Q:

$$\begin{aligned} dv_t &= k_v^Q(m_t - v_t)dt + \sigma_v\sqrt{v_t}\,dW_{2t} \\ dm_t &= k_m^Q(\theta_m^Q - m_t)dt + \sigma_m\sqrt{m_t}\,dW_{3t} \end{aligned}$$

The following relationship holds between P and Q parameters:

$$\begin{aligned} k_v^P &= k_v^Q - \gamma_2\sigma_v \\ k_m^P &= k_m^Q - \gamma_3\sigma_m \\ \theta_m^P &= \theta_m^Q\, k_m^Q/k_m^P \end{aligned}$$

According our estimates γ_2 and γ_3 are both negative.

B.3.1 Theoretical $\alpha(\tau)$ and $\beta(\tau)$

In the Expectation Hypothesis for the stochastic jump-intensity two-factor model, the following definitions and quantities are used in the calculation of theoretical $\alpha(\tau)$ and $\beta(\tau)$:

$$\begin{aligned} \mathrm{QV}_{t,t+\tau} &= \frac{1}{\tau}\int_t^{t+\tau} v_s\,ds + \frac{1}{\tau}\sum_{N_t\le u\le N_T} J_u^2 =: \mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau} \\ E^P[\mathrm{QV}_{t,t+\tau}] &= \theta_v^P + E^P[J^2](\lambda_0 + \lambda_1\theta_v^P) \\ E_t^P[\mathrm{QV}^c_{t,t+\tau}] &= v_t\phi_v^P(\tau) + \frac{k_v^Q}{k_v^P}m_t\phi_m^P(\tau) + \frac{k_v^Q}{k_v^P}\theta_m^P(1 - \phi_v^P(\tau) - \phi_m^P(\tau)) \\ E_t^P[\mathrm{QV}^j_{t,t+\tau}] &= E^P[J^2](\lambda_0 + \lambda_1 E_t^P[\mathrm{QV}^c_{t,t+\tau}]) \\ E^P[\mathrm{QV}^j_{t,t+\tau}] &= E^P[J^2](\lambda_0 + \lambda_1\theta_v^P) \\ \mathrm{VS}_{t,t+\tau} &= E^Q[J^2]\lambda_0 + [1 + \lambda_1 E^Q[J^2]][v_t\phi_v^Q(\tau) + m_t\phi_m^Q(\tau) + \theta_m^Q(1 - \phi_v^Q(\tau) - \phi_m^Q(\tau))] \\ E^P[\mathrm{VS}_{t,t+\tau}] &= E^Q[J^2]\lambda_0 + [1 + \lambda_1 E^Q[J^2]][\theta_v^P\phi_v^Q(\tau) + \theta_m^P\phi_m^Q(\tau) + \theta_m^Q(1 - \phi_v^Q(\tau) - \phi_m^Q(\tau))] \\ E_t^P[y_t - E^P[y_t]] &= E_t^P[\mathrm{QV}_{t,t+\tau} - E^P[\mathrm{QV}_{t,t+\tau}]] \\ &= E_t^P[\mathrm{QV}^c_{t,t+\tau} - \theta_v^P] + E_t^P[\mathrm{QV}^j_{t,t+\tau} - E^P[J^2](\lambda_0 + \lambda_1\theta_v^P)] \\ &= \phi_v^P(\tau)(v_t - \theta_v^P) + \frac{k_v^Q}{k_v^P}\phi_m^P(\tau)(m_t - \theta_m^P) + E^P[J^2]\lambda_1(E_t^P[\mathrm{QV}^c_{t,t+\tau} - \theta_v^P]) \end{aligned}$$

$$
\begin{aligned}
&= [1+\lambda_1 E^P[J^2]][\phi_v^P(\tau)(v_t-\theta_v^P)+\frac{k_v^Q}{k_v^P}\phi_m^P(\tau)(m_t-\theta_m^P)] \\
x_t - E[x_t] &= \mathrm{VS}_{t,t+\tau} - E^P[\mathrm{VS}_{t,t+\tau}] \\
&= [1+\lambda_1 E^Q[J^2]][\phi_v^Q(\tau)(v_t-\theta_v^P)+\phi_m^Q(\tau)(m_t-\theta_m^P)]
\end{aligned}
$$

and using these expressions

$$
\begin{aligned}
\mathrm{Cov}^P[x_t,y_t] &= E^P\{(x_t-E^P[x_t])(y_t-E^P[y_t])\} = E^P\{(x_t-E^P[x_t])\times E_t^P[y_t-E^P[y_t]]\} \\
&= E^P\{([1+\lambda_1 E^Q[J^2]][\phi_v^Q(\tau)(v_t-\theta_v^P)+\phi_m^Q(\tau)(m_t-\theta_m^P)])\times \\
&\quad [1+\lambda_1 E^P[J^2]][\phi_v^P(\tau)(v_t-\theta_v^P)+\frac{k_v^Q}{k_v^P}\phi_m^P(\tau)(m_t-\theta_m^P)]\} \\
&= (1+\lambda_1 E^Q[J^2])\,[1+\lambda_1 E^P[J^2]]\{\phi_v^Q(\tau)\phi_v^P(\tau)\mathrm{Var}^P[v_t]+\phi_m^Q(\tau)\frac{k_v^Q}{k_v^P}\phi_m^P(\tau)\mathrm{Var}^P[m_t] \\
&\quad +\left(\phi_v^Q(\tau)\frac{k_v^Q}{k_v^P}\phi_m^P(\tau)+\phi_m^Q(\tau)\phi_v^P(\tau)\right)\mathrm{Cov}^P[v_t,m_t]\} \\
&=: (1+\lambda_1 E^Q[J^2])\,\mathrm{Cov}[x_t,y_t]^{twofactor} \\
\mathrm{Var}^P[x_t] &= (1+\lambda_1 E^Q[J^2])^2\,[\phi_v^Q(\tau)^2\mathrm{Var}^P[v_t]+\phi_m^Q(\tau)^2\mathrm{Var}^P[m_t]+2\phi_v^Q(\tau)\phi_m^Q(\tau)\mathrm{Cov}^P[v_t,m_t]] \\
&=: (1+\lambda_1 E^Q[J^2])^2\,\mathrm{Var}^P[x_t]^{twofactor}
\end{aligned}
$$

with obvious notation for $\mathrm{Cov}[x_t,y_t]^{twofactor}$ and $\mathrm{Var}^P[x_t]^{twofactor}$. Hence

$$
\begin{aligned}
\beta(\tau) &= \frac{\mathrm{Cov}^P[x_t,y_t]}{\mathrm{Var}^P[x_t]} = \frac{(1+\lambda_1 E^P[J^2])}{(1+\lambda_1 E^Q[J^2])}\frac{\mathrm{Cov}^P[x_t,y_t]^{twofactor}}{\mathrm{Var}^P[x_t]^{twofactor}} = \frac{(1+\lambda_1 E^P[J^2])}{(1+\lambda_1 E^Q[J^2])}\,\beta(\tau)^{twofactor} \\
\alpha(\tau) &= E[y_t]-\beta(\tau)\times E[x_t] \\
&= (\theta_v^P+E^P[J^2](\lambda_0+\lambda_1\theta_v^P))-\beta(\tau)\times \\
&\quad \{E^Q[J^2]\lambda_0+[1+\lambda_1 E^Q[J^2]][\theta_v^P\phi_v^Q(\tau)+\theta_m^P\phi_m^Q(\tau)+\theta_m^Q(1-\phi_v^Q(\tau)-\phi_m^Q(\tau))]\}
\end{aligned}
$$

Note that λ_0 enters in $\alpha(\tau)$, but not $\beta(\tau)$. Hence $\beta(\tau)$ is the same under the two-factor and constant jump-intensity two-factor model (where $\lambda_1 = 0$).

As expected, setting $\lambda_0 = \lambda_1 = 0$ and $\sigma_m = 0$ implies $k_m^P = k_m^Q$ which implies $\theta_m^P = \theta_m^Q =: \theta^Q$ which implies $m_t = \theta^Q$ for all t which implies $\mathrm{Var}^P[m_t] = \mathrm{Cov}^P[m_t, v_t] = 0$, we recover $\beta(\tau)$ and $\alpha(\tau)$ in the Heston model. Limits $\alpha(\tau)$ and $\beta(\tau)$ when $\tau \to 0$ or $\tau \to \infty$ can be easily calculated using expressions above.

Recall

$$
\lim_{\tau\to 0}\phi_v(\tau)=1,\quad \lim_{\tau\to 0}\phi_m(\tau)=0,\quad \lim_{\tau\to\infty}\phi_v(\tau)=0,\quad \lim_{\tau\to\infty}\phi_m(\tau)=0
$$

As for β:

$$
\lim_{\tau\to 0}\beta(\tau)^{twofactor} = 1
$$

$$
\begin{aligned}
\lim_{\tau\to\infty}\beta(\tau)^{twofactor} &= \frac{k_m^Q k_v^Q (k_m^P k_m^Q \mathrm{Var}^P[v_t] + (k_v^Q)^2 \mathrm{Var}^P[m_t] + (k_m^P + k_m^Q) k_v^Q \mathrm{Cov}^P[v_t, m_t])}{k_m^P k_v^P ((k_m^Q)^2 \mathrm{Var}^P[v_t] + (k_v^Q)^2 \mathrm{Var}^P[m_t] + 2k_m^Q k_v^Q \mathrm{Cov}^P[v_t, m_t])} \\
\lim_{\tau\to 0}\beta(\tau) &= \frac{(1+\lambda_1 E^P[J^2])}{[1+\lambda_1 E^Q[J^2]]} \\
\lim_{\tau\to\infty}\beta(\tau) &= \frac{(1+\lambda_1 E^P[J^2])}{[1+\lambda_1 E^Q[J^2]]}\lim_{\tau\to\infty}\beta(\tau)^{twofactor}
\end{aligned}
$$

Notice that stochastic jump intensity ($\lambda_1 \neq 0$) and jump risk premium ($E^P[J^2] \neq E^Q[J^2]$) implies $\lim_{\tau\to\infty}\beta(\tau) \neq 1$.

When $\mathrm{Var}^P[m_t] = \mathrm{Cov}^P[v_t, m_t] = 0$, $\lim_{\tau\to\infty}\beta(\tau)^{twofactor} = k_v^Q/k_v^P$ as in the Heston model.

As for α:

$$
\begin{aligned}
\lim_{\tau\to 0}\alpha(\tau) &= (\theta_v^P + E^P[J^2](\lambda_0 + \lambda_1\theta_v^P)) - \frac{(1+\lambda_1 E^P[J^2])}{(1+\lambda_1 E^Q[J^2])}[E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2])\theta_v^P] \\
&= \frac{\lambda_0(E^P[J^2] - E^Q[J^2])}{1+\lambda_1 E^Q[J^2]} \\
\lim_{\tau\to\infty}\alpha(\tau) &= [\theta_v^P + E^P[J^2](\lambda_0 + \lambda_1\theta_v^P)] - \lim_{\tau\to\infty}\beta(\tau)[E^Q[J^2]\lambda_0 + [1+\lambda_1 E^Q[J^2]]\theta_m^Q]
\end{aligned}
$$

B.3.2 Theoretical $R^2(\tau)$

To calculate the theoretical $R^2(\tau)$ we need to calculate $\mathrm{Var}^P[y_t]$. Recall

$$
y_t = \frac{1}{\tau}\int_t^{t+\tau} v_s\, ds + \frac{1}{\tau}\sum_{u=N_t}^{N_{t+\tau}} J_u^2 := \mathrm{QV}_{t,t+\tau}^c + \mathrm{QV}_{t,t+\tau}^j
$$

To calculate $\mathrm{Var}^P[\mathrm{QV}_{t,t+\tau}^c]$ the following expectations are used. Recall that under P, $dv_t = k_v^P(k_v^Q/k_v^P m_t - v_t)dt + \sigma_v\sqrt{v_t}\, dW_{2t}$, where $k_v^Q = k_v^P + \gamma_2\sigma_v$. When $u \leq s$

$$
\begin{aligned}
E^P[v_u\, v_s] &= E^P[v_u\, E^P[v_s|v_u, m_u]] \\
&= E^P\Bigg[v_u \Bigg\{ v_u\, e^{-k_v^P(s-u)} + \frac{k_v^Q}{k_v^P} m_u \left[k_v^P \frac{e^{-k_m^P(s-u)} - e^{-k_v^P(s-u)}}{k_v^P - k_m^P}\right] \\
&\quad + \frac{k_v^Q}{k_v^P}\theta_m^P \left[1 + \frac{k_m^P e^{-k_v^P(s-u)} - k_v^P e^{-k_m^P(s-u)}}{k_v^P - k_m^P}\right]\Bigg\}\Bigg] \\
&= E^P[v_u^2] e^{-k_v^P(s-u)} + E^P[v_u m_u]\frac{k_v^Q}{k_v^P}\left[k_v^P \frac{e^{-k_m^P(s-u)} - e^{-k_v^P(s-u)}}{k_v^P - k_m^P}\right] \\
&\quad + E^P[v_u]\theta_m^P \frac{k_v^Q}{k_v^P}\left[1 + \frac{k_m^P e^{-k_v^P(s-u)} - k_v^P e^{-k_m^P(s-u)}}{k_v^P - k_m^P}\right] \\
E^P[v_u\, m_u] &= E^P[E^P[v_u\, m_u|m_u]] = E^P[(k_v^Q/k_v^P)\, m_u\, m_u] = \frac{k_v^Q}{k_v^P}E^P[m_u^2] \\
E^P[m_u^2] &= \mathrm{Var}^P[m_u] + (E^P[m_u])^2 = \frac{\sigma_m^2\theta_m^P}{2k_m^P} + (\theta_m^P)^2.
\end{aligned}
$$

Plugging $E^P[v_u\, v_s]$ above into the double integral below gives, after some calculations,

$$\begin{aligned}
\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] &= \frac{2}{\tau^2}\int_t^{t+\tau} ds \int_t^s du\ E^P[v_u\, v_s] - (\theta_v^P)^2 \\
&= \frac{2}{\tau^2(k_v^P - k_m^P)}\left\{ -k_v^Q \frac{(1-e^{-k_m^P \tau})\mathrm{Cov}^P[v,m]}{(k_m^P)^2} \right. \\
&\quad \left. + \frac{(1-e^{-k_v^P \tau})((k_v^P - k_m^P)(-\mathrm{Var}^P[v]) + k_v^Q \mathrm{Cov}^P[v,m])}{(k_v^P)^2} \right\} \\
&\quad + \frac{2}{\tau}\, \frac{k_v^Q \mathrm{Cov}^P[v,m] + k_m^P \mathrm{Var}^P[v]}{k_m^P k_v^P}
\end{aligned}$$

When $m_t = \theta_m^P$ for all t, which implies $\mathrm{Cov}^P[v,m] = 0$, the expression above gives the variance of annualized integrated variance for the Heston model.

To calculate $\mathrm{Var}^P[\mathrm{QV}_{t,t+\tau}^j]$ the following calculations are used. Let $\overline{\mathrm{QV}}_{t,t+\tau}^c = \int_t^{t+\tau} v_s\, ds$ denote the non-annualized continuous part of the quadratic variation, and similarly $\overline{\mathrm{QV}}_{t,t+\tau}^j$ for the jump part. Conditional on $\mathrm{QV}_{t,t+\tau}^c$, the counting process $(N_{t+\tau} - N_t)$ has inhomogeneous Poisson distribution which implies

$$E^P[(N_{t+\tau} - N_t)|\mathrm{QV}_{t,t+\tau}^c] = \mathrm{Var}^P[(N_{t+\tau} - N_t)|\mathrm{QV}_{t,t+\tau}^c] = \lambda_0 \tau + \lambda_1 \overline{\mathrm{QV}}_{t,t+\tau}^c$$

The following calculations and definitions lead to the analytic expression for $\mathrm{Var}^P[y_t]$, omitting subscript $_{t,t+\tau}$,

$$\begin{aligned}
\overline{\mathrm{QV}}^j &= \sum_{u=N_t}^{N_{t+\tau}} J_u^2 = J_{N_t}^2 + \ldots + J_{N_{t+\tau}}^2 \\
E^P[\mathrm{QV}^c] &= \theta_v^P \\
E^P[(\mathrm{QV}^c)^2] &= \mathrm{Var}^P[\mathrm{QV}^c] + (\theta_v^P)^2 \\
E^P[\overline{\mathrm{QV}}^j|\mathrm{QV}^c] &= E^P[J^2](\lambda_0\, \tau + \lambda_1 \overline{\mathrm{QV}}^c) \\
E^P[\mathrm{QV}^j] &= E^P[J^2](\lambda_0 + \lambda_1 E^P[\mathrm{QV}^c]) = E^P[J^2](\lambda_0 + \lambda_1 \theta_v^P) \\
\mathrm{Var}^P[\overline{\mathrm{QV}}^j|\mathrm{QV}^c] &= \mathrm{Var}^P\{E^P[\overline{\mathrm{QV}}^j|N_{t+\tau}, \mathrm{QV}^c]|\mathrm{QV}^c\} + E^P\{\mathrm{Var}[\overline{\mathrm{QV}}^j|N_{t+\tau}, \mathrm{QV}^c]|\mathrm{QV}^c\} \\
&= \mathrm{Var}\{(N_{t+\tau} - N_t)E^P[J^2]|\mathrm{QV}^c\} + E^P\{(N_{t+\tau} - N_t)\mathrm{Var}^P[J^2]|\mathrm{QV}^c\} \\
&= (E^P[J^2])^2(\lambda_0 \tau + \lambda_1 \overline{\mathrm{QV}}^c) + \mathrm{Var}^P[J^2](\lambda_0 \tau + \lambda_1 \overline{\mathrm{QV}}^c) \\
&= E^P[J^4](\lambda_0 \tau + \lambda_1 \overline{\mathrm{QV}}^c) \\
\mathrm{Var}^P[\overline{\mathrm{QV}}^j] &= \mathrm{Var}^P\{E^P[\overline{\mathrm{QV}}^j|\mathrm{QV}^c]\} + E^P\{\mathrm{Var}[\overline{\mathrm{QV}}^j|\mathrm{QV}^c]\} \\
&= \mathrm{Var}^P\{E^P[J^2](\lambda_0\, \tau + \lambda_1 \overline{\mathrm{QV}}^c)\} + E^P\{E^P[J^4](\lambda_0 \tau + \lambda_1 \overline{\mathrm{QV}}^c)\} \\
&= (E^P[J^2])^2 \lambda_1^2 \mathrm{Var}\{\overline{\mathrm{QV}}^c\} + E^P[J^4](\lambda_0 \tau + \lambda_1 E^P[\overline{\mathrm{QV}}^c])\}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}^P[\mathrm{QV}^j] &= \frac{1}{\tau^2}\mathrm{Var}^P[\overline{\mathrm{QV}}^j] = (E^P[J^2])^2\lambda_1^2\mathrm{Var}^P\{\mathrm{QV}^c\} + E^P[J^4](\frac{\lambda_0}{\tau} + \frac{\lambda_1}{\tau}E^P[\mathrm{QV}^c])\} \\
E^P[J^4] &= (\mu_j^P)^4 + 6(\mu_j^P)^2\sigma_j^2 + 3\sigma_j^4, \qquad J \sim \mathcal{N}(\mu_j^P, \sigma_j^2) \\
E^P[\mathrm{QV}^c\,\mathrm{QV}^j] &= E^P[J^2][\lambda_0 E^P[\mathrm{QV}^c] + \lambda_1 E^P[(\mathrm{QV}^c)^2]] = E^P[J^2][\lambda_0\theta_v^P + \lambda_1 E^P[(\mathrm{QV}^c)^2]] \\
\mathrm{Cov}^P[\mathrm{QV}^c, \mathrm{QV}^j] &= E^P[\mathrm{QV}^c\mathrm{QV}^j] - E^P[\mathrm{QV}^c]\,E^P[\mathrm{QV}^j] \\
&= E^P[J^2](\lambda_0\theta_v^P + \lambda_1 E^P[(\mathrm{QV}^c)^2]) - \theta_v^P\,E^P[J^2](\lambda_0 + \lambda_1 E^P[\mathrm{QV}^c]) \\
&= E^P[J^2]\lambda_1 E^P[(\mathrm{QV}^c)^2] - \theta_v^P\,E^P[J^2]\lambda_1\theta_v^P = E^P[J^2]\lambda_1\mathrm{Var}[\mathrm{QV}^c] \\
\mathrm{Var}^P[y_t] &= \mathrm{Var}^P[\mathrm{QV}^c] + \mathrm{Var}^P[\mathrm{QV}^j] + 2\mathrm{Cov}^P[\mathrm{QV}^c, \mathrm{QV}^j]
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathrm{Var}[\frac{1}{\tau}(N_{t+\tau} - N_t)] &= \mathrm{Var}\,E[\frac{1}{\tau}(N_{t+\tau} - N_t)|\mathrm{QV}^c_{t,t+\tau}] + E\,\mathrm{Var}[\frac{1}{\tau}(N_{t+\tau} - N_t)|\mathrm{QV}^c_{t,t+\tau}] \\
&= \mathrm{Var}[\lambda_0 + \lambda_1\mathrm{QV}^c_{t,t+\tau}] + \frac{1}{\tau^2}E[\lambda_0\tau + \lambda_1\overline{\mathrm{QV}}^c_{t,t+\tau}] \\
&= \lambda_1^2\mathrm{Var}[\mathrm{QV}^c_{t,t+\tau}] + \frac{\lambda_0}{\tau} + \frac{\lambda_1}{\tau}E[\mathrm{QV}^c_{t,t+\tau}]
\end{aligned}
$$

hence $\mathrm{Var}[\frac{1}{\tau}(N_{t+\tau} - N_t)] \to 0$ when $\tau \to \infty$.

References

Aït-Sahalia, Y., 1999, "Transition Densities for Interest Rate and Other Nonlinear Diffusions," *Journal of Finance*, 54, 1361–1395.

——— , 2002a, "Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach," *Econometrica*, 70, 223–262.

——— , 2002b, "Telling from Discrete Data Whether the Underlying Continuous-Time Model is a Diffusion," *Journal of Finance*, 57, 2075–2112.

——— , 2008, "Closed-Form Likelihood Expansions for Multivariate Diffusions," *Annals of Statistics*, 36, 906–937.

Aït-Sahalia, Y., and R. Kimmel, 2007, "Maximum Likelihood Estimation of Stochastic Volatility Models," *Journal of Financial Economics*, 83, 413–452.

——— , 2010, "Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions," *Journal of Financial Economics*, 98, 113144.

Aït-Sahalia, Y., and P. A. Mykland, 2003, "The Effects of Random and Discrete Sampling When Estimating Continuous-Time Diffusions," *Econometrica*, 71, 483–549.

Alizadeh, S., M. Brandt, and F. Diebold, 2002, "Range-Based Estimation of Stochastic Volatility Models," *Journal of Finance*, 57, 1047–1091.

Amengual, D., 2009, "The Term Structure of Variance Risk Premia," working paper, Princeton University.

Andersen, T., L. Benzoni, and J. Lund, 2002, "An Empirical Investigation of Continuous-Time Equity Return Models," *Journal of Finance*, 57, 1239–1284.

Bates, D. S., 2000, "Post-'87 Crash Fears in the S&P 500 Futures Option Market," *Journal of Econometrics*, 94, 181–238.

Bollerslev, T., and V. Todorov, 2011, "Tails, Fears, and Risk Premia," *Journal of Finance*, forthcoming.

Britten-Jones, M., and A. Neuberger, 2000, "Option Prices, Implied Price Process, and Stochastic Volatility," *Journal of Finance*, 55, 839–866.

Broadie, M., M. Chernov, and M. Johannes, 2007, "Model Specification and Risk Premia: Evidence from Futures Options," *Journal of Finance*, 62, 1453–1490.

——— , 2009, "Understanding Index Options Returns," *Review of Financial Studies*, 22, 4493–4529.

Broadie, M., and A. Jain, 2008, "The Effect of Jumps and Discrete Sampling on Volatility and Variance Swaps," *International Journal of Theoretical and Applied Finance*, 11, 761–797.

Carr, P., and R. Lee, 2010, "Hedging Variance Options on Continuous Semimartingales," *Finance and Stochastics*, forthcoming.

Carr, P., and D. Madan, 1998, "Towards a Theory of Volatility Trading," in *Risk Book on Volatility*, ed. by R. Jarrow. New York: Risk, pp. 417–427.

Carr, P., and L. Wu, 2003, "What Type of Process Underlies Options? A Simple Robust Test," *Journal of Finance*, 58, 2581–2610.

——— , 2006, "A Tale of Two Indices," *Journal of Derivatives*, 13, 13–29.

——— , 2009, "Variance Risk Premiums," *Review of Financial Studies*, 22, 1311–1341.

Chernov, M., R. Gallant, E. Ghysels, and G. Tauchen, 2003, "Alternative Models of Stock Price Dynamics," *Journal of Econometrics*, 116, 225 – 257.

Chernov, M., and E. Ghysels, 2000, "A Study Towards a Unified Approach to the Joint Estimation of Objective and Risk Neutral Measures for the Purpose of Options Valuation," *Journal of Financial Economics*, 57, 407–458.

Duffie, D., J. Pan, and K. J. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343–1376.

Egloff, D., M. Leippold, and L. Wu, 2010, "The Term Structure of Variance Swap Rates and Optimal Variance Swap Investments," *Journal of Financial and Quantitative Analysis*, 45, 1279–1310.

Eraker, B., 2004, "Do Equity Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices," *Journal of Finance*, 59, 1367–1403.

Eraker, B., M. S. Johannes, and N. Polson, 2003, "The Impact of Jumps in Equity Index Volatility and Returns," *Journal of Finance*, 58, 1269–1300.

Filipović, D., E. Gourier, and L. Mancini, 2011, "Quadratic Variance Swap Models," working paper, EPFL.

Fuertes, C., and A. Papanicolaou, 2011, "Implied Probability Measures of Volatility," working paper, Princeton University.

Gangahar, A., 2006, "Volatility Becomes an Asset Class," *Financial Times*, May 23.

Jacod, J., and P. Protter, 1998, "Asymptotic Error Distributions for the Euler Method for Stochastic Differential Equations," *Annals of Probability*, 26, 267–307.

Jarrow, R., Y. Kchia, M. Larsson, and P. Protter, 2011, "Discretely Sampled Variance and Volatility Swaps Versus Their Continuous Approximations," working paper, Cornell University.

Jensen, B., and R. Poulsen, 2002, "Transition Densities of Diffusion Processes: Numerical Comparison of Approximation Techniques," *Journal of Derivatives*, 9, 18–32.

Jiang, G. J., and Y. S. Tian, 2005, "The Model-Free Implied Volatility and its Information Content," *Review of Financial Studies*, 18, 1305–1342.

Jung, J., 2006, "Vexed by Variance," *Risk Magazine*, 19(8), August.

Martin, I., 2011, "Simple Variance Swaps," working paper, Stanford University.

Naik, V., and M. Lee, 1990, "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies*, 3, 493–521.

Pan, J., 2002, "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journal of Financial Economics*, 63, 3–50.

Protter, P., 2005, *Stochastic Integration and Differential Equations.* Springer-Verlag, New York, second edn.

Schultes, R., 2008, "Volatility Kills Market for Volatility Swaps," *Wall Street Journal*, October 22.

Schwert, G. W., 1987, "Effects of model specification on tests for unit roots in macroeconomic data," *Journal of Monetary Economics*, 20, 73–103.

Todorov, V., 2010, "Variance Risk Premium Dynamics: The Role of Jumps," *Review of Financial Studies*, 23, 345–383.

van Binsbergen, J. H., W. H. Hueskes, R. S. J. Koijen, and E. B. Vrugt, 2011, "Equity Yields," working paper, NBER.

Whaley, R. E., 2000, "The Investor Fear Gauge," *The Journal of Portfolio Management*, 26, 12–17.

Yu, J., 2007, "Closed-Form Likelihood Approximation and Estimation of Jump-Diffusion with an Application to the Realignment Risk of the Chinese Yuan," *Journal of Econometrics*, 141, 1245–1280.

Part IV

Euler Approximation and Likelihood Expansion for Continuous-Time Derivative Pricing Models: A Comparative Analysis

Mustafa Karaman[1]

Abstract

An extensive Monte Carlo study is performed to compare the estimation performance of Euler discretization and closed-form likelihood expansion (LE) methodologies for four nested derivative pricing models, the most general of which is the stochastic intensity jump diffusion two-factor stochastic volatility model. The estimation of models with real data is also performed using variance swap (VS) rates. The results show that LE produces lower standard errors than Euler approximation. Moreover, both methods generate similar point estimates when models are not too far away from the multivariate normal density. However, when there are jumps in the stock price dynamic then those methods are producing diverging results with LE's superiority. Therefore, for more realistic derivative pricing models which feature a jump component, LE should be the method of estimation.

[1]I thank EPFL for providing the computing resources at Greedy Pool.

1. Introduction

Estimation of the continuous-time models in finance is essential whenever those models are used for derivatives pricing purposes, portfolio allocation decisions, risk management intentions, or even taking "bet" type of statistical arbitrage positions. However this estimation possesses some challenges because the continuous-time data cannot be observed in the market even at the highest frequency available. In the literature there is a big number of contributions to tackle with this issue via for example, generalized method of moments, nonparametric techniques, or simulation based estimations (see, Aït-Sahalia (1999) and the references therein).

Whenever applicable maximum likelihood estimation (MLE) makes the best use of data in hand in the sense that the estimates would be consistent and the standard errors of the estimates reach the Cramer-Rao Lower Bound. Therefore, MLE produces (asymptotically) efficient estimates. The basic requirement of the MLE on the other hand is that the transition density of the process should be known for time horizons larger than infinitesimal distance, e.g. daily, weekly or monthly, since the data is available for such time intervals. Models in Black and Scholes (1973), Vasicek (1977), Cox, Ingersoll, and Ross (1985), and Cox (1975) all have known closed-form transition densities.

Most of the recent continuous-time derivatives pricing models in finance literature this density is unknown in closed-form and needs to be approximated. In this paper, I perform this approximation by Euler discretization and LE where the first one is basically the normal density, and the second one is a "deformed" or "stretched" normal density. In LE, by stretching the normal density, the dynamics-implied skewness, kurtosis and the higher moments are aimed to be captured.

In the next section, I present the most general model that is developed in the previous part of the dissertation and show how it nests four different derivatives pricing models that are commonly used in finance. Using those models, I perform an extensive simulation exercise and study the estimation performance with simulated data where I know precisely the data generating process. The simulation study shows that filtering latent variables from observed VS data has no significant impact on the estimation performance. Moreover, it also shows that although Euler discretization is sufficiently accurate for models not far away from the multivariate normal density for the daily time interval, its accuracy is decreasing while the dynamics of the state variables become more complex. The real data estimation part, where I use VS data, also reveals that LE outperforms the Euler approximation while the dynamics of the state variables include jumps. That is, when the transition density of the state vector is further away from the multivariate normal density, the method of estimation should be LE. Finally, the cross-comparison of the model estimates shows that the appropriate parametric model for VS data is the stochastic-intensity two-factor volatility model.

2. The Model

In this section I briefly develop stochastic intensity two-factor volatility model which is presented Aït-Sahalia, Karaman, and Mancini (2012). I start with historical, P-dynamics and then move into risk-neutral, Q-dynamics by assuming no-arbitrage and specifying the market prices of risks. Then, I show how this model nests, Heston, Two-Factor Volatility (without jumps) and Constant Intensity Two-Factor Volatility Models.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions (see Protter (2005)), with P denoting the objective probability measure. Let S be a semimartingale modeling the stock price or index process, v be the stochastic volatility[1] and m be a Feller's square root process modeling the long-run mean of the volatility, v, with the following dynamics

$$\begin{aligned} dS_t/S_{t_-} &= \mu_t dt + \sqrt{(1-\rho^2)v_t}\, dW_{1t}^P + \rho\sqrt{v_t}\, dW_{2t}^P + (\exp(J_t^P) - 1)\, dN_t - \nu_t^P\, dt \\ dv_t &= k_v^P(m_t\, k_v^Q/k_v^P - v_t)dt + \sigma_v \sqrt{v_t}\, dW_{2t}^P \\ dm_t &= k_m^P(\theta_m^P - m_t)dt + \sigma_m\sqrt{m_t}\, dW_{3t}^P \end{aligned} \tag{1}$$

where $\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t$, all Brownian increments are uncorrelated, ρ is the instantaneous correlation between stock returns and spot variance, $k_v^Q = k_v^P + \gamma_2\sigma_v$, and γ_2 determines the market price of risk for W_{2t}^P. The spot variance, v_t, follows a two-factor model with $m_t\, k_v^Q/k_v^P$ being the stochastic long-run mean level, in turn m_t mean reverts to θ_m^P, and k_v^P and k_m^P are positive speed of mean reversions. Using stock and option prices, previous empirical work documented that two factors are necessary to describe stochastic volatility dynamics; see e.g. Andersen, Benzoni, and Lund (2002), Alizadeh, Brandt, and Diebold (2002) and Chernov, Gallant, Ghysels, and Tauchen (2003). One factor is fast mean reverting and volatile to capture sudden movements in volatility, the other factor is persistent and less volatile to capture long run movements in volatility. The square-root specification of the diffusion parts is adopted to keep the model close to commonly adopted specifications (e.g. Todorov (2010)). The methods developed below do not require such specification. For example $\sqrt{v_t}$ could be replaced by, say,

[1]Note that it is indeed stochastic variance, however I follow the convention to call it volatility.

$v_t^{\gamma_v}$ with γ_v being one more parameter to be estimated. The level of variance swap rates will not be affected by such specification, but of course the dynamic is. The random jump size J_t^P is independent of the filtration generated by the Brownian motions and jump process, and normally distributed with mean μ_j^P and variance σ_j^2, hence $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$, as for instance in Bates (2000) and Pan (2002). g^Q is similarly defined with μ_j^P replaced by μ_j^Q. The jump intensity is $\lambda_t = \lambda_0 + \lambda_1 v_t$, where λ_0 and λ_1 are positive constants. This specification allows more price jumps to occur during more volatile periods, as often observed empirically, with the intensity bounded away from 0 by λ_0.

At this stage I specify the market price of risks for the Brownian motions as

$$\Lambda_t = [\gamma_1\sqrt{(1-\rho^2)v_t}, \quad \gamma_2\sqrt{v_t}, \quad \gamma_3\sqrt{m_t}]'.$$

The jump size risk premium is $(g^P - g^Q)$ where $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$ and g^Q is similarly defined. Hence $(g^P - g^Q)(\lambda_0 + \lambda_1 v_t)$ is the time-varying (total) jump risk premium.

P- and Q-Brownian motions are related as follows:

$$\begin{aligned} dW_{1t}^P &= dW_{1t}^Q - \gamma_1\sqrt{(1-\rho^2)v_t}\,dt \\ dW_{2t}^P &= dW_{2t}^Q - \gamma_2\sqrt{v_t}\,dt \\ dW_{3t}^P &= dW_{3t}^Q - \gamma_3\sqrt{m_t}\,dt. \end{aligned}$$

The relationship between P and Q parameters are

$$\begin{aligned} k_v^P &= k_v^Q - \gamma_2\sigma_v \\ k_m^P &= k_m^Q - \gamma_3\sigma_m \\ \theta_m^P &= \theta_m^Q\, k_m^Q/k_m^P \end{aligned}$$

As in Pan (2002) I assume that the jump intensity is the same under both measures and only the average jump size changes from μ_j^P to μ_j^Q when changing measure from P to Q. This assumption implies that all the jump risk premium is absorbed by the jump size risk premium, $(g^P - g^Q)$. Given typical sample sizes, accurate estimation of risk premiums for both jump-size and jump-timing is obviously challenging and I will not relax this assumption in this paper.

The jump component in the stock price makes the market incomplete with respect to the risk-free bank account, the stock and any finite number of derivatives. Hence, the state price density is not unique [2].

[2] See Aït-Sahalia, Karaman, and Mancini (2012) for a possible specification of the state price density

Given the market prices of risks above, under Q the ex-dividend price process follows

$$
\begin{aligned}
dS_t/S_{t_-} &= (r-\delta)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^Q + \rho\sqrt{v_t}\,dW_{2t}^Q + (\exp(J_t^Q)-1)\,dN_t^Q - \nu_t^Q\,dt \\
dv_t &= k_v^Q(m_t - v_t)dt + \sigma_v\sqrt{v_t}\,dW_{2t}^Q \\
dm_t &= k_m^Q(\theta_m^Q - m_t)dt + \sigma_m\sqrt{m_t}\,dW_{3t}^Q
\end{aligned} \tag{2}
$$

where r is the risk-free rate, δ the dividend yield (both taken to be constant for simplicity only), and the Brownian motions W_1^Q and W_2^Q, jump size J^Q, jump process N^Q, and its compensator ν^Q are all governed by the measure Q. When no confusion arises superscripts P and Q are omitted.

In the parametric model above I assume the existence of an equivalent risk-neutral measure Q. By the Fundamental Theorem of Asset Pricing (see Delbaen and Schachermayer (1994)), absence of arbitrage is therefore maintained and VS rates can be calculated easily. By convention the variance swap contract has zero value at inception. Assuming that the interest rate does not depend on the quadratic variation, no arbitrage implies

$$
\mathrm{VS}_{t,t+\tau} = E_t^Q[\mathrm{QV}_{t,t+\tau}] \tag{3}
$$

where E_t^Q denotes time-t conditional expectation under Q, and $\mathrm{QV}_{t,t+\tau}$ is the annualized quadratic variation of the log-price process under Q-dynamics. Interchanging expectation and integration (justified by Fubini's theorem), and exploiting independence between J^Q and N^Q

$$
\begin{aligned}
\mathrm{VS}_{t,t+\tau} &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q[v_s]ds + \frac{1}{\tau}E^Q[J^2]\,E_t^Q[N_{t+\tau}-N_t] \\
&= E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2])[(1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_v^Q(\tau)v_t + \phi_m^Q(\tau)m_t]
\end{aligned} \tag{4}
$$

where $E^Q[J^2] = E_t^Q[J^2]$ as random jump size is time-homogeneous, and

$$
\begin{aligned}
\phi_v^Q(\tau) &= (1-\exp(-k_v^Q\tau))/(k_v^Q\tau) \\
\phi_m^Q(\tau) &= \left(1+\exp(-k_v^Q\tau)k_m^Q/(k_v^Q-k_m^Q) - \exp(-k_m^Q\tau)k_v^Q/(k_v^Q-k_m^Q)\right)/(k_m^Q\tau).
\end{aligned}
$$

Given the linearity of the variance swap payoff in the spot variance, only the drift of v_t enters the variance swap rate. The diffusion part of v_t (or volatility of volatility) affects only the dynamic of $\mathrm{VS}_{t,t+\tau}$.[3] The Q-expectation of squared jump size, $E^Q[J^2]$, provides constant contribution to the variance swap rate (independent of the time to maturity), while the stochastic intensity provides time-varying contribution to $\mathrm{VS}_{t,t+\tau}$.

[3] This can easily be seen by specifying some dynamic for the spot variance, such as Heston model, calculating $\int_t^{t+\tau} E_t^Q[v_s]ds$ explicitly and then applying Itô's formula to it.

In the simulation and real-data analysis, the model in (1) is reduced to Heston, Two-Factor Volatility (without jumps) and Constant Intensity Two-Factor Volatility Model via the restrictions below:

1. Heston model, obtained setting $\lambda_0 = \lambda_1 = 0$ and $m_t = \theta_v^P$ for all t in (1)

2. Two-Factor Volatility model, obtained setting $\lambda_0 = \lambda_1 = 0$ in (1)

3. Constant jump-intensity two-factor model, obtained setting $\lambda_1 = 0$ in (1)

4. Stochastic jump-intensity two-factor model as in (1).

Restricting the variance dynamic in (2) to the Heston model by setting $m_t = \theta_v^Q$ for all t and $\lambda_0 = \lambda_1 = 0$, the VS rate becomes $\mathrm{VS}_{t,t+\tau} = (1 - \phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)v_t$, i.e. a weighted average between v_t and θ_v^Q. When $\tau \to 0$, $\phi_v^Q(\tau) \to 1$ and $\mathrm{VS}_{t,T} \to v_t$. Eliminating the jump component, imposing the restriction $\lambda_0 = \lambda_1 = 0$ for all t in (2), the VS rate is immediately seen by plugging this restriction into (4). VS rates can now exhibit various degrees of persistence and volatility and the induced term structure can take a variety of shapes depending on the relative levels of v_t, m_t and θ_m^Q. This two-factor model can reproduce, at least qualitatively, previous feature of VS rates but cannot accommodate the priced jump component in VS rates. Egloff, Leippold, and Wu (2010) study VS rates under the previous diffusion one- and two-factor stochastic volatility model.

3. Estimation Method

In order to estimate the model in (1) and (2) I need time series data for the underlying S&P 500 index and at least two more time series to recover the unobserved volatility and the long-run mean of the volatility. The key feature of stochastic volatility models with affine drift is that model-based VS rates are affine in latent state variables. This feature suggests to filter out latent states directly from VS rates. Then I use likelihood-based methods namely Euler approximation and LE to estimate the models. The state vector follows a multivariate jump-diffusion stochastic volatility process and thus its transition density is unknown. Since jumps in stock prices are rare events, using Bayes rule I approximate the transition density of the state vector with a mixture of no-jump and 1-jump. Such densities are unknown as well but can be approximated. I approximate the transition density in case of no-jump first by Euler approximation and second by LE methods. The latter was introduced and developed in the univariate setting by Aït-Sahalia (2002a), Aït-Sahalia (1999), and extended to the multivariate setting by Aït-Sahalia (2008), Aït-Sahalia and Kimmel (2007), Aït-Sahalia and Kimmel (2010), Aït-Sahalia, Karaman, and Mancini (2012). The reason of focusing on these two estimation methodologies is to compare the one of the sophisticated methodologies (i.e. the LE) with the one of the most easily applicable techniques of estimation (i.e. Euler approximation).

The transition density p_X can be written by the Bayes' formula as follows

$$p_X(x_\Delta|x_0) = p_X(x_\Delta|x_0, N_\Delta = 0)\Pr(N_\Delta = 0) + p_X(x_\Delta|x_0, N_\Delta = 1)\Pr(N_\Delta = 1) + o(\Delta)$$

where $\Pr(N_\Delta = j)$ is the probability that j jumps occur at day Δ, omitting the dependence on Θ for brevity. In Model (1)–(2), the largest contribution to the transition density of X (hence to the likelihood) comes from the first term when conditioning on no jump occurring during Δ, i.e. $N_\Delta = 0$. The probability of such event, $\Pr(N_\Delta = 0)$, is large and of the order $1 - (\lambda_0 + \lambda_1 v_0)\,\Delta$. The contribution of the second term is only of the order $(\lambda_0 + \lambda_1 v_0)\,\Delta$. As in the setting Δ is one day, the contribution of higher order terms appear to be quite modest. The advantage of this approximation is that the leading term $p_X(x_\Delta|x_0, N_\Delta = 0)$ can be approximated by both Euler approximation and the LE methods.

4. Monte Carlo Simulation

I run a Monte Carlo simulation to check the accuracy of the two estimation methods, namely Euler and LE. I simulate four models

1. Heston model, obtained setting $\lambda_0 = \lambda_1 = 0$ and $m_t = \theta_v^P$ for all t in (1)
2. Two-Factor Volatility model, obtained setting $\lambda_0 = \lambda_1 = 0$ in (1)
3. Constant jump-intensity two-factor model, obtained setting $\lambda_1 = 0$ in (1)
4. Stochastic jump-intensity two-factor model as in (1).

Each model nests its preceding models, with the last one being the most general model considered here. The simulated sample size is $N = 10000$ data points sampled at daily frequency ($\Delta = 1/252$). Each day interval is divided into 30 sub-intervals, which corresponds to roughly a 15-minute interval. States variables are simulated at such intraday frequency using an Euler discretization of the corresponding dynamics. In practice, on each simulated day, 29 out 30 simulated data points are discarded to obtain the daily sample of S_{t_i}, v_{t_i}, and m_{t_i}. VS rates are calculated for various maturities using simulated values and the various models. Each simulated path is initialized with variance and its stochastic long run mean at their unconditional means and the stock price at 100. To reduce the impact of such initial values on the simulated trajectory, an initial 500 values are generated and then discarded, taking the last one as the starting point for the sample trajectory. For each model I simulate 1000 trajectories. Since latent variables are known in the simulation exercise, I can study the accuracy of the various estimation methods when using directly the latent variables (though they are unobservable in practice) and when recovering them from simulated VS rates, hence quantifying the impact of filtering latent states.

Overall, simulation results suggest that likelihood-based methods are all fairly accurate, although there are some differences across various methods as discussed below.

4.1. Heston Model

Tables 1 and 2 show simulation results for the Heston model using latent variance and filtered variance via 3 months to maturity VS rates, respectively, as well as simulated underlying stock returns. Both LE and Euler methods deliver fairly accurate results. LE produces slightly better results in terms of bias and standard errors of the estimates for most parameters. The reason of this similar performance is that for the daily interval the Euler approximation works well for those models with transition density not too far from the multivariate normal density. Comparing the two tables shows that the impact on estimates of recovering latent variance from observed variance swap rates is small. Aït-Sahalia and Kimmel (2007) report a similar simulation study but when latent variance is recovered via European options. The deterioration of estimation results seems to be less pronounced when using VS, consistent with the feature that VS rates have a more direct link to latent variance than European options.

4.2. Two-Factor Stochastic Volatility Model

Tables 3 and 4 report simulation results for the two-factor model using latent variance and its stochastic long run mean and 3-month and 12-month to maturity VS rates, respectively, as well as stock returns. Again, both LE and Euler methods deliver fairly accurate results. However, compared to the Heston model, the first method produces relatively more accurate results than the second one, as transition densities of state variables are now more far away from normality. Similar to the case with Heston model, the deterioration of estimation results when recovering latent state variables from observed VS rates is negligible.

At this stage we see that using underlying stock returns and observed VS rates, instead of latent variables directly, has no impact on estimation performance. Moreover we also realize that the power of LE is getting superior as the dynamics of the state variables move further away the multivariate normal density (see also Jensen and Poulsen (2002)). For this reason in the rest of the simulations I only analyze the LE technology using simulated variance swap rates and the index.

4.3. Constant Jump Intensity Two-Factor Stochastic Volatility Model

Table 5 reports simulation results for the constant jump-intensity two-factor model using LE technique. Latent state variables are filtered using 3- and 12-month to maturity VS rates. Despite the presence of jumps in stock price, the LE method provides quite accurate estimation results for most of the parameters. However the market price of risks, the speed of mean reversion and jump intensity parameters posses some difficulties in estimation as they have relatively higher mean bias and root mean squared error.

4.4. Stochastic Jump Intensity Two-Factor Stochastic Volatility Model

Table 6 report simulation results for the the stochastic jump-intensity two-factor model using LE methodology. The simulation setting is the same as in the previous model. Again, the method is fairly accurate for most of the parameters while estimating market price of risks, the speed of mean reversion and jump parameters is a bit more challenging.

5. Estimation with Variance Swap Data

In this section I report parameter estimates of the four models discussed above, namely Heston, two-factor volatility, constant jump-intensity, and stochastic jump-intensity models, using the likelihood-based procedures namely Euler, and LE methods using VS data. As each model nests preceding models, I can assess empirically whether each extra layer of flexibility is actually relevant for fitting data and economically important for generating risk premiums. Since there are more VS rates than latent state variables, for each model I perform 3 estimations for each method considered:

i) Only ℓ VS rates assumed to be observed without error (and used to recover the ℓ latent state variables, and $\ell = 1$ or 2 depending on the model);

ii) All 5 VS rates of which ℓ are observed without error and $5-\ell$ are observed with uncorrelated errors;
iii) All 5 VS rates of which ℓ are observed without error and $5-\ell$ are observed with potentially correlated errors.
The error is defined as observed VS rate minus model-based rate, see Aït-Sahalia, Karaman, and Mancini (2012) for further details. In the Tables 8 - 11, I report these estimates under the columns "VS without errors", "VS with uncorrelated errors", and "VS with correlated errors", respectively.

5.1. *Dataset*

I estimate the four nested models listed above by using over the counter quotes on variance swap rates on the S&P 500 index and the index itself. The database, whose descriptive statistics can be seen at Table 7, is provided by a major broker dealer in New York City. The data are daily closing quotes on variance swap rates with fixed time to maturities at two, three, six, twelve, and twenty four months from January 4, 1996 to September 2, 2010. There are $3,624$ observations for each maturity.
The in sample period where I perform the estimations of the models is from January 4, 1996 to April 2, 2007. Therefore the out of sample period is from April 3, 2007 to September 2, 2010.

5.2. *Heston Model*

Table 8 reports parameter estimates for the Heston model. In this model the latent state variable is the spot variance only, hence $\ell = 1$. Euler and LE method provide quite similar point estimates, especially for the correlated errors case that is the case where I make use of all available data. However, standard errors for the second method tend to be smaller. The long-run mean volatility, $\sqrt{\theta_v^P}$, is firmly around 20%, and the correlation, ρ, around -70%. As it is usually the case, market price of risk parameters are estimated less precisely, but γ_2 is always negative implying a negative variance risk premium, see also Tables 12 - 13. The mean reversion speed parameter, k_v^P, changes significantly from "VS without errors" to "VS with uncorrelated errors" estimations, and implied half-life[4] ranges from about 90 to 190 days for the estimates. The correlation parameter for error terms, ρ_e, is positive suggesting that the Heston model is systematically underpricing or overpricing variance swaps.

[4]Half-life is defined as the time necessary to halve a unit shock and here is given by $-\log(0.5)/k_v^P \times 252$ in number of days.

5.3. *Two-Factor Stochastic Volatility Model*

Table 9 reports parameter estimates for the two-factor model based on Euler and LE methods. All parameter estimates are quite inline with each other, especially in the last two sets of estimates, i.e. last 8 columns. However in terms of standard errors LE is superior. The long-run average volatility, $\sqrt{\theta_m^P}$, is around 23%. The estimates of correlation parameter, ρ is around -74% and both γ_2 and γ_3 are negative implying negative variance risk premium, see also Tables 14 - 15. Compared to the Heston model, estimates of k_v^P are larger, implying much faster mean reversion of spot variance, while estimates of k_m^P are about 1/20 of k_v^P implying a much slower mean reversion of stochastic long-run mean. These estimates suggest that the two-factor model releases the "tension" in k_v^P in Heston model. The correlation parameter for error terms, ρ_e, is slightly negative[5] suggesting that the two-factor model fits variance swap rates better than the Heston model without any systematic pricing error.

5.4. *Constant Jump Intensity Two-Factor Stochastic Volatility Model*

Table 10 shows estimation results for constant jump-intensity two-factor model using Euler and LE methods. The point estimates of this model are not always inline with each other. In particular the jump and market prices of risk parameters are estimated quite differently via both methods. In terms of standard errors on the other hand LE is performing better. The long-run average volatility is around 22%. The estimates of correlation parameter, ρ is around -72% and both γ_2 and γ_3 are negative implying negative variance risk premium, see Tables 16 - 17. The expected jump size is positive under the objective probability measure, $\mu_j^P > 0$, and negative under the risk neutral measure, $\mu_j^Q < 0$, which implies a positive jump risk premium. The correlation parameter estimated for error terms, ρ_e, is negative suggesting that the model has no systematic pricing error.

5.5. *Stochastic Jump Intensity Two-Factor Stochastic Volatility Model*

Table 11 shows estimation results for stochastic-intensity two-factor model using Euler and LE methods. Similar to the constant intensity jumps case, there are slight differences in the point estimation results of the parameters like mean-reversion speed, market prices of risks and jump parameters. However still the standard errors are less in case of LE for most parameters. The

[5]The determinant of the 3×3 error term correlation matrix is $2\rho_e^3 - 3\rho_e^2 + 1$ which is strictly positive as long as $\rho_e > -0.5$.

long-run average volatility is around 21%. All market price of Brownian risks are negative, the equity risk premium which is sum of diffusive and jump risk premiums is positive and the variance risk premium is negative, see Tables 18 - 19. Expected jump size is positive under the objective measure, μ_j^P, and negative under the risk neutral measure, μ_j^Q, implying positive jump risk premium[6]. The correlation parameter for error terms, ρ_e, is negative suggesting that the model does not make any systematic pricing error.

5.6. *Comparison Across Different Models Estimates*

For each model, estimation based on all available variance swap rates has significantly higher log-likelihood when compared to other estimations. Moreover, LE is producing less standard errors for the estimates. Therefore, the analysis below will be based on such estimates. The Heston Model has low log-likelihood compared to other models and systematically misprices VS rates, confirming that the model is misspecified. Excluding the Heston model, all other model estimates based on "VS with uncorrelated errors" and "VS with correlated errors" are close in the sense that they imply similar dynamics of variance, stochastic long-run mean and recovered instantaneous risk premiums, see Figures 1 - 8 and Tables 12 - 19. Moreover, standard deviations and correlations of model-based error terms are also similar. Focusing on the last two models with jumps they both have quite similar mean and volatility of jump size estimates. Estimates of jump intensity implies on average 5 jumps per year for both models. Finally comparing log-likelihoods of last two models with jumps suggests that the last model is significantly better. The gain in log-likelihood of having a stochastic jump intensity parameter in the form of $\lambda_0 + \lambda_1 v_t$ is large.

5.7. *Pricing Error of Variance Swap Rates*

Pricing error of VS rate is defined as model-based VS rate, $\widehat{\text{VS}}$, minus actual VS rate. To save space I report in- and out-sample pricing errors only for Heston and stochastic jump-intensity two-factor models in Tables 20 - 23 also see Figures 9 - 12. Pricing error for the other models are available upon request. The stochastic jump-intensity two-factor model fits VS rates well both in- and out-sample and significantly outperforms the Heston model for example in terms of root mean square error (RMSE).

[6]Except for the Euler with the correlated pricing errors case. However this still implies positive jump risk premium.

6. Conclusions

In this paper I compare the estimation performance of Euler approximation and LE. I study four different derivatives pricing models that are commonly used in finance. In the first part of the paper, I perform an extensive simulation study and show that filtering latent variables from observed VS data has no significant impact on the estimation performance. Moreover, I also show that although Euler discretization is sufficiently accurate for models not far away from the multivariate normal density for the daily time interval, its accuracy is decreasing while the dynamics of the state variables become more complex. For the real data estimation I use VS rates written on S&P 500 index quoted by a major broker dealer. This part also reveals that LE outperforms the Euler approximation while the dynamics of the state variables include jumps. That is, when the transition density of the state vector is further away from the multivariate normal density, the method of estimation should be LE. Finally, the cross-comparison of the model estimates shows that the appropriate parametric model for VS data is the stochastic-intensity two-factor volatility model.

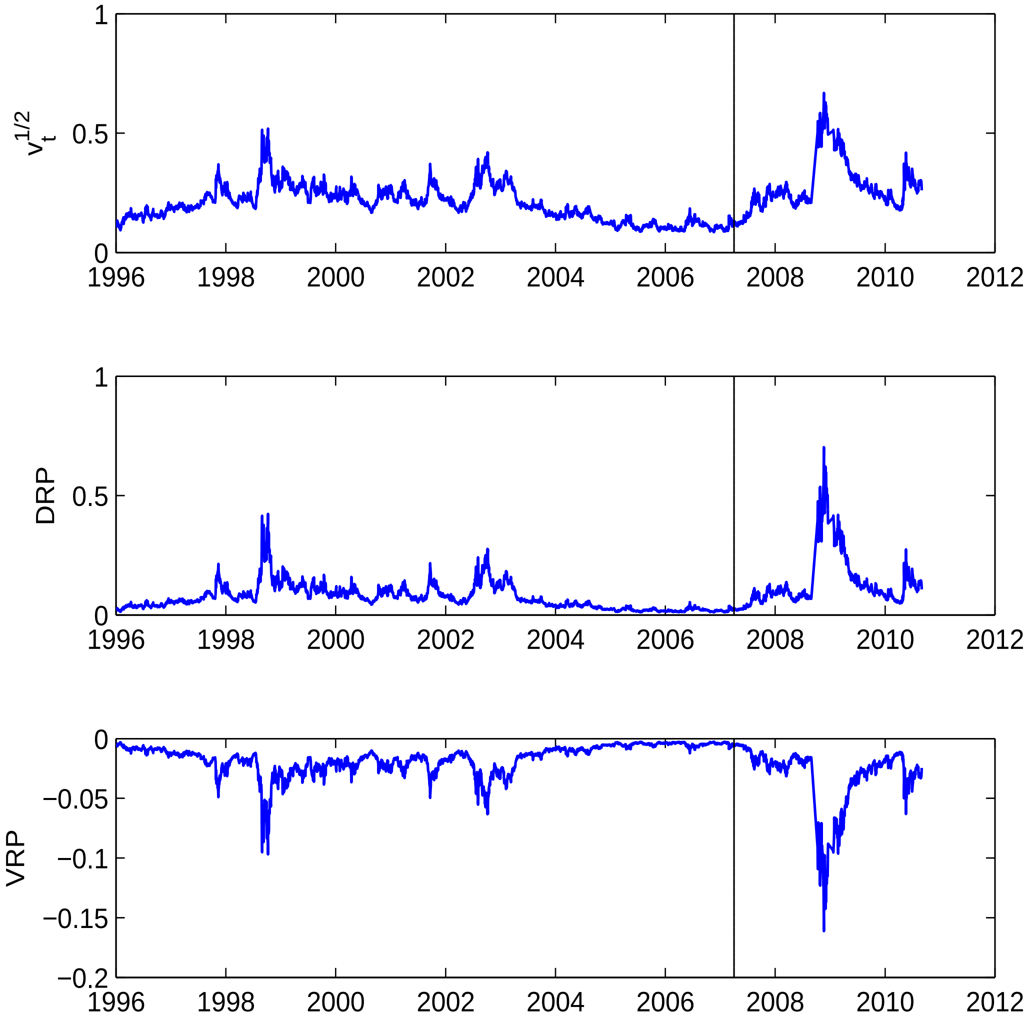

Figure 1. Instantaneous Recovered Quantities for Heston Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$.

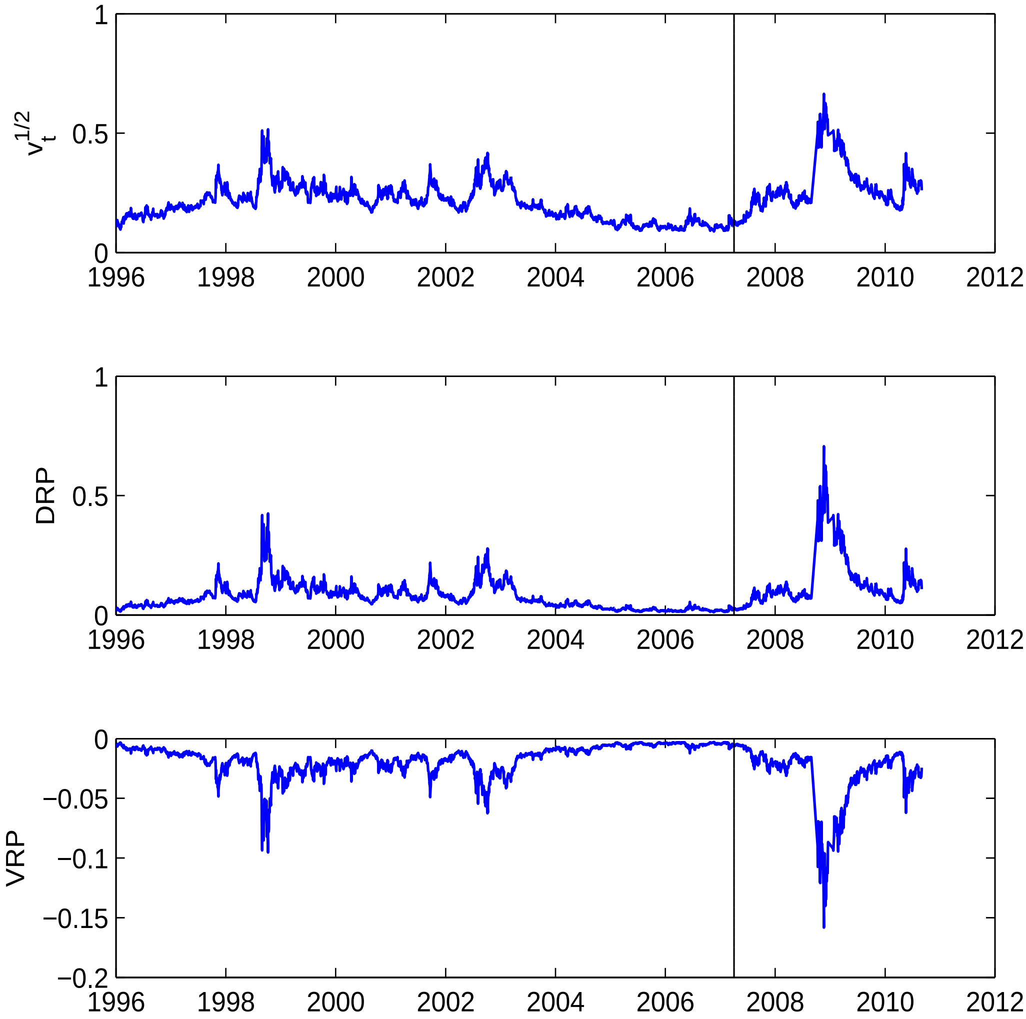

Figure 2. Instantaneous Recovered Quantities for Heston Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$.

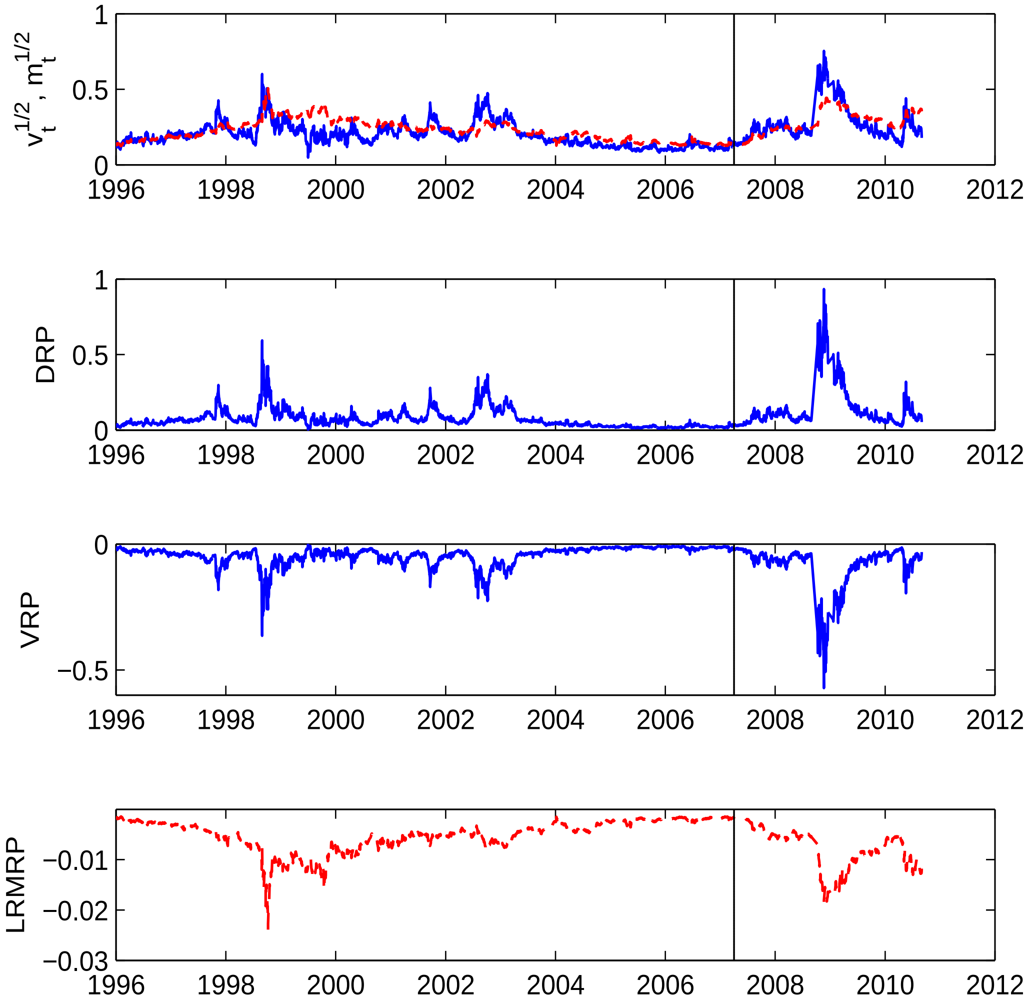

Figure 3. Instantaneous Recovered Quantities for 2-Factor Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

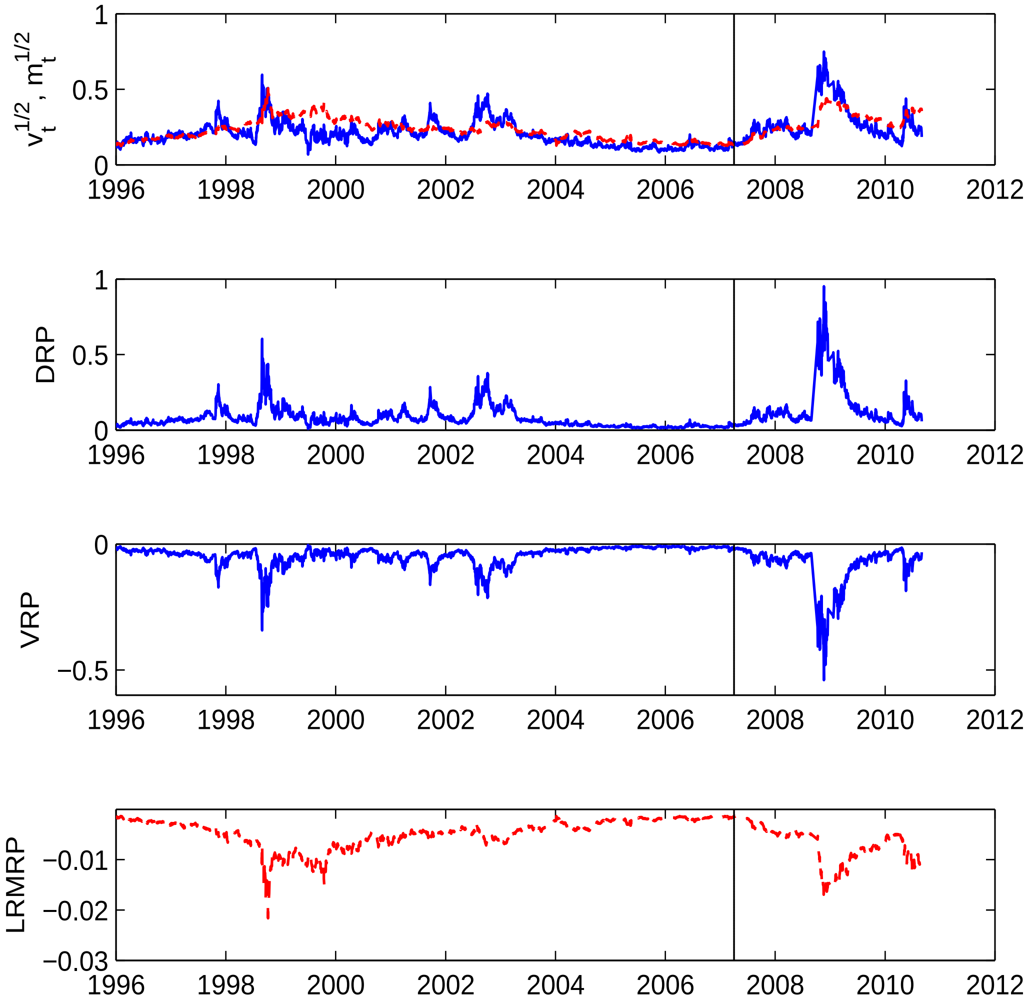

Figure 4. Instantaneous Recovered Quantities for 2-Factor Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\mathrm{LRMRP}_t = \gamma_3\sigma_m m_t$.

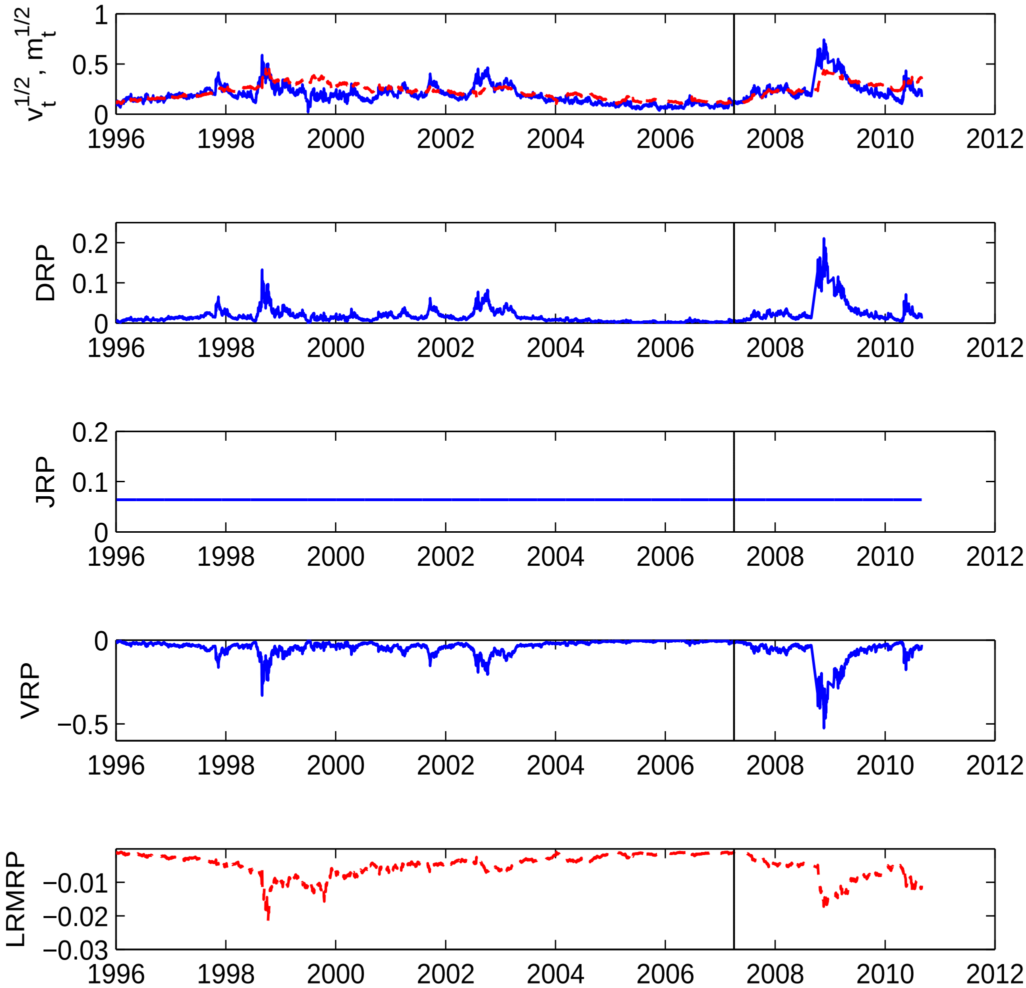

Figure 5. Instantaneous Recovered Quantities for Constant Intensity Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J]-E^Q[e^J])(\lambda_0)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

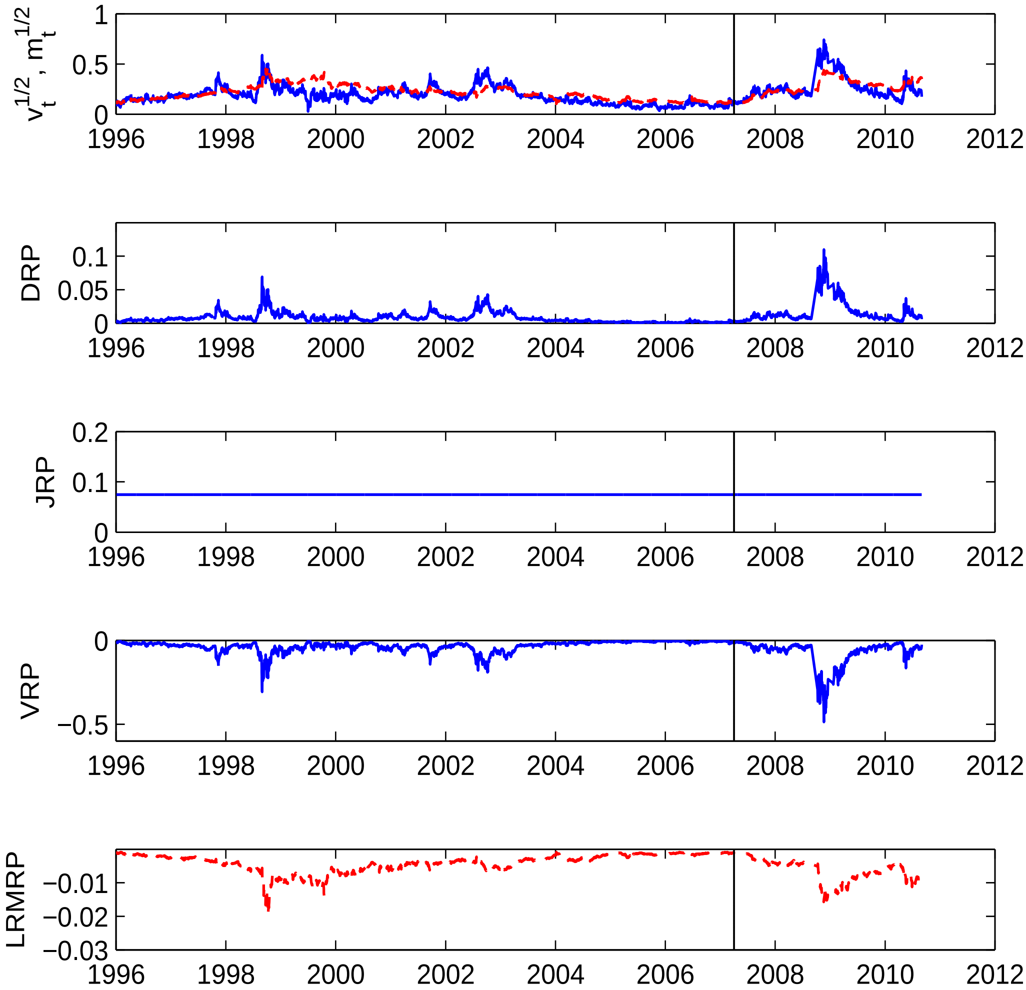

Figure 6. Instantaneous Recovered Quantities for Constant Intensity Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J]-E^Q[e^J])(\lambda_0)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

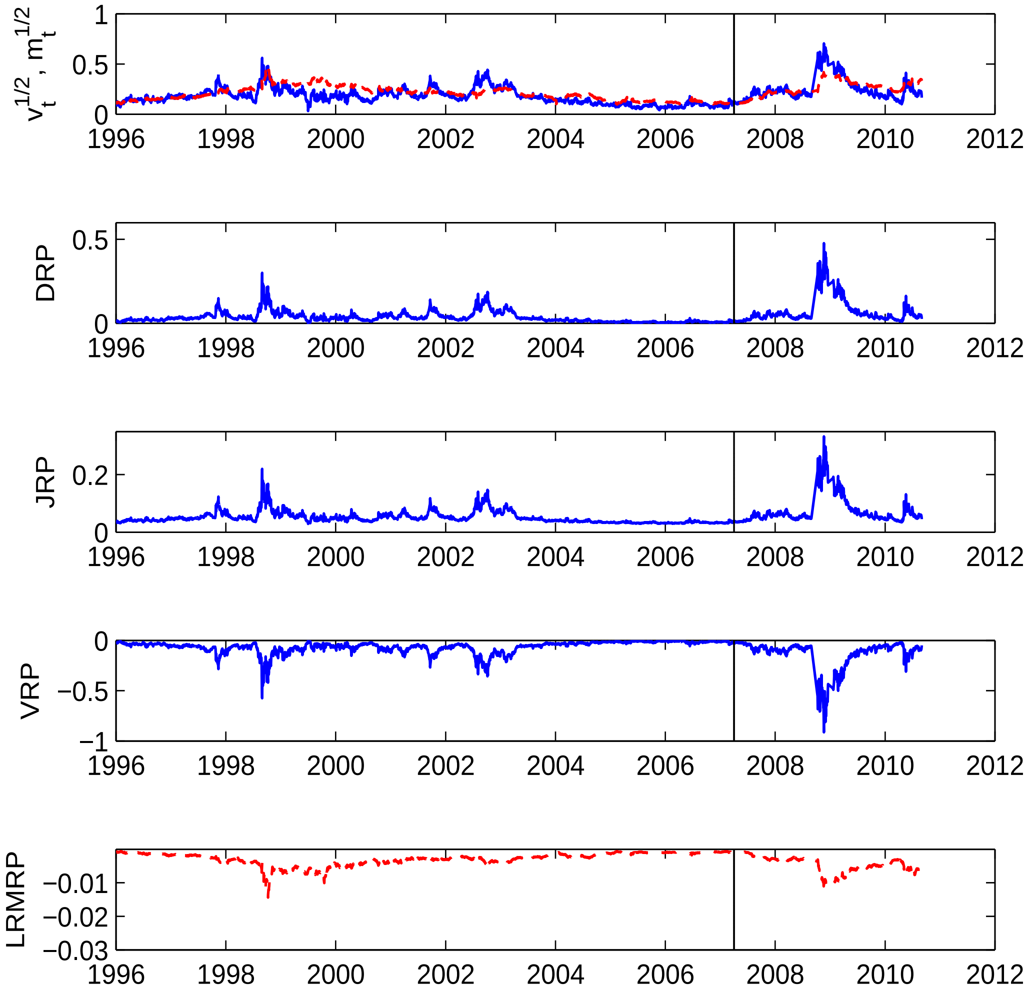

Figure 7. Instantaneous Recovered Quantities for Stochastic Intensity Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\mathrm{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0+\lambda_1 v_t)$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\mathrm{LRMRP}_t = \gamma_3\sigma_m m_t$.

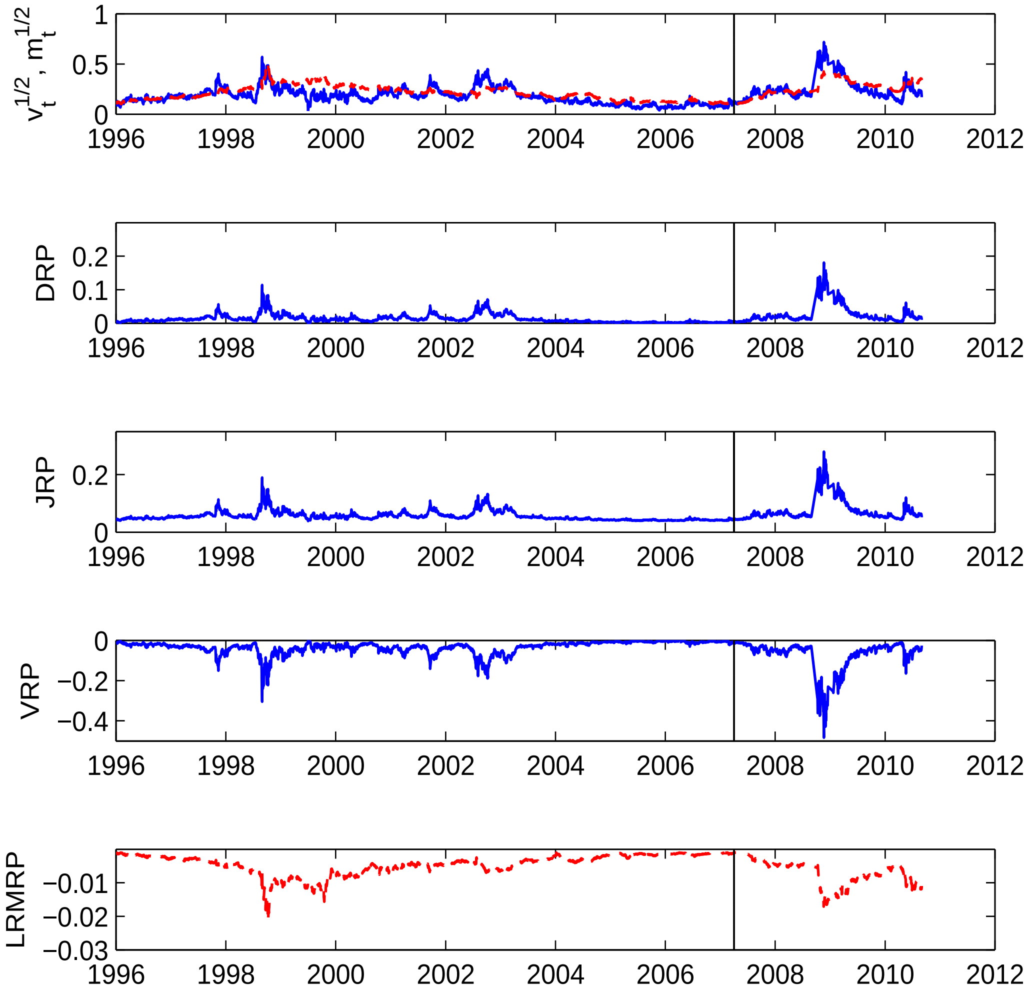

Figure 8. Instantaneous Recovered Quantities for Stochastic Intensity Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007. Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0 + \lambda_1 v_t)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

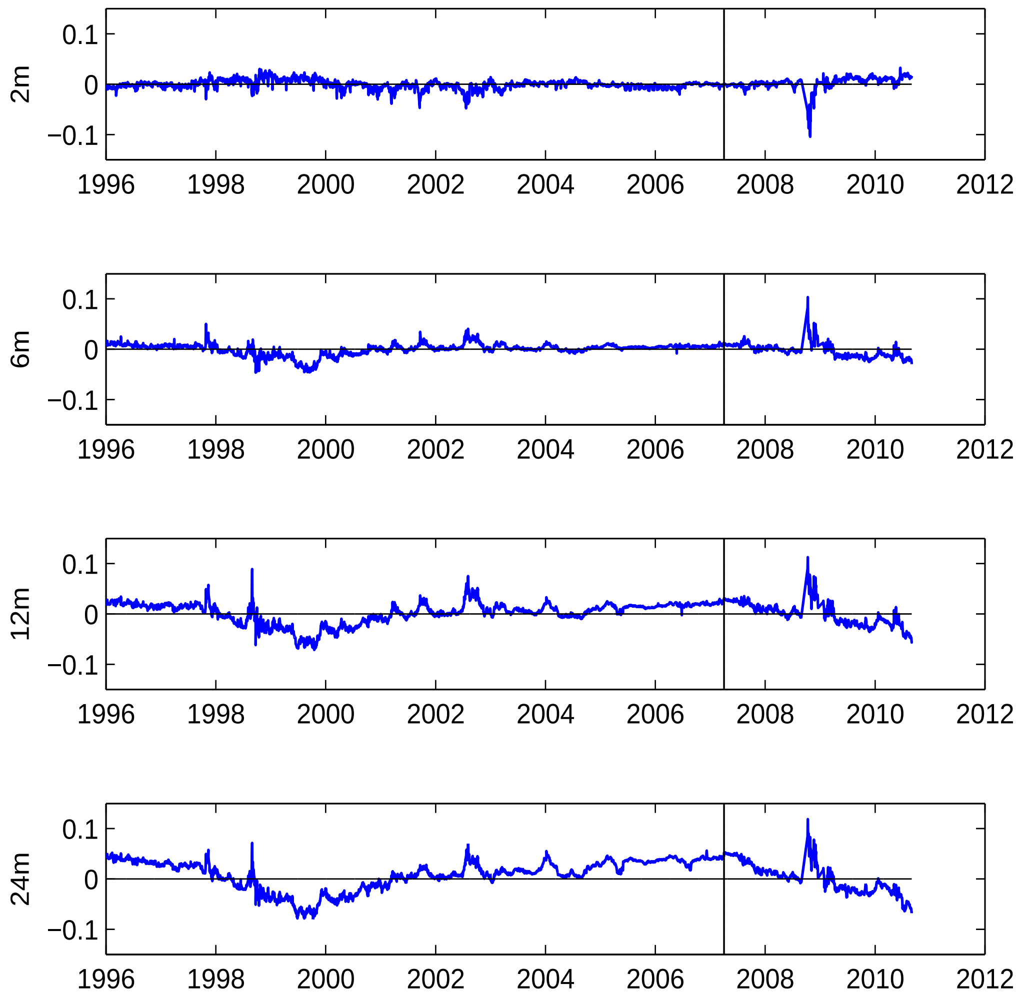

Figure 9. Difference between Recovered and Quoted Variance Swap Rates (in volatility units) for Heston Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

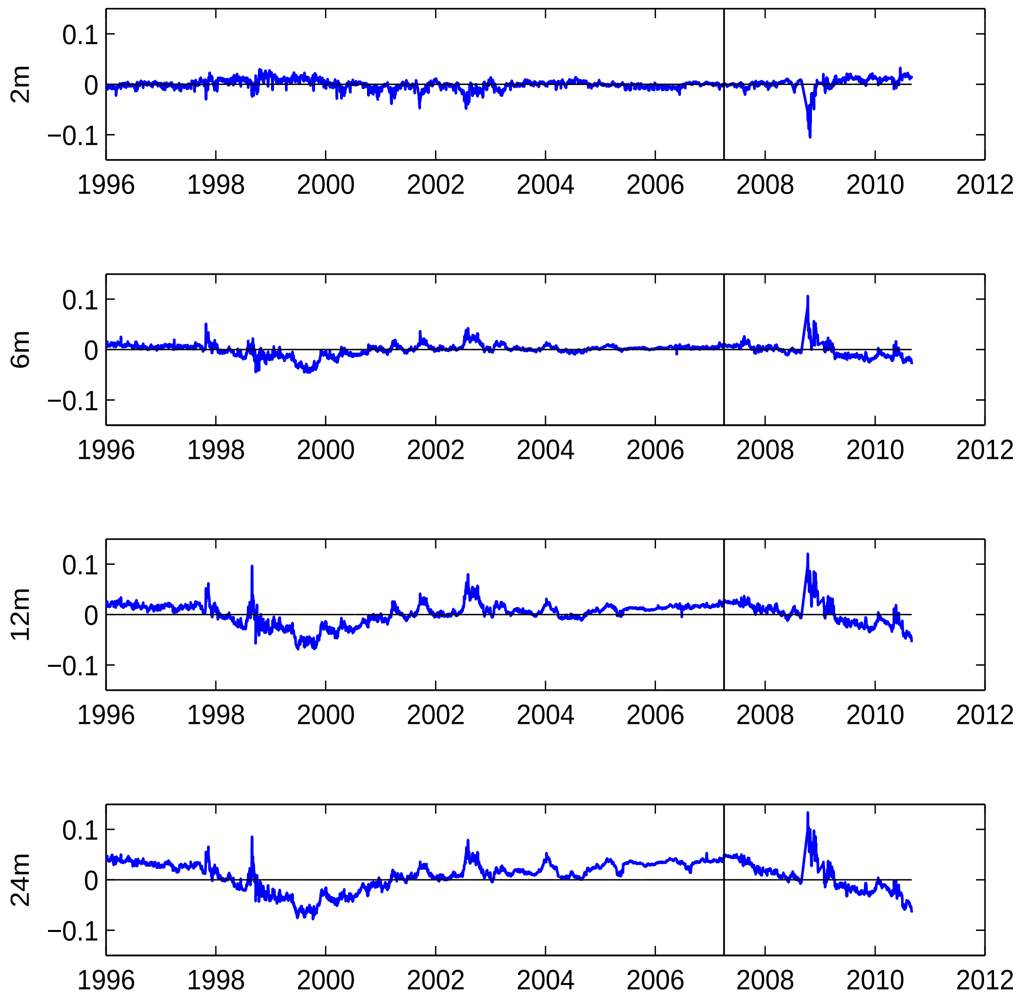

Figure 10. Difference between Recovered and Quoted Variance Swap Rates (in volatility units) for Heston Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

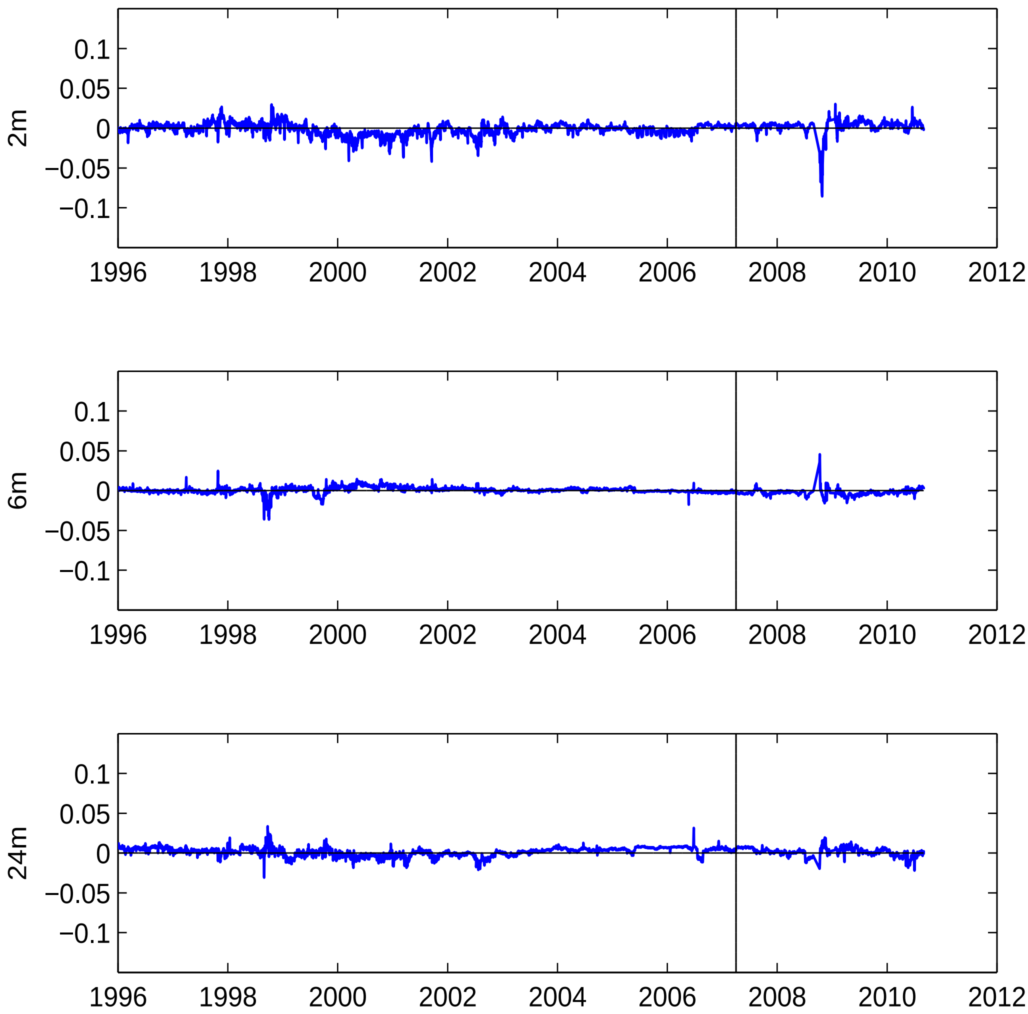

Figure 11. Difference between Recovered and Quoted Variance Swap Rates (in volatility units) for stochastic intensity JD 2F-SV Model - Euler. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

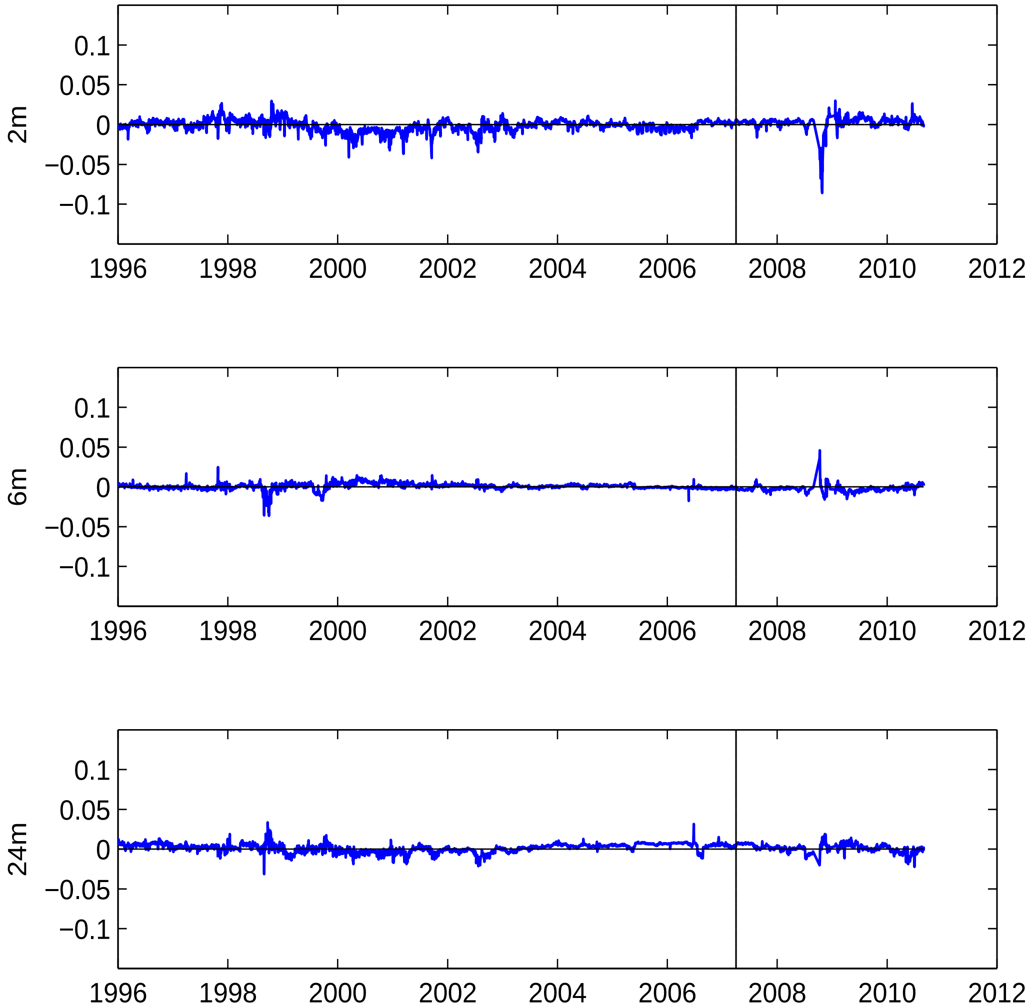

Figure 12. Difference between Recovered and Quoted Variance Swap Rates (in volatility units) for stochastic intensity JD 2F-SV Model - LE. Vertical line denotes beginning of out-sample period, i.e., April 3, 2007.

Table 1. Monte Carlo Simulation Heston Model: estimation results using stock prices and latent variance ($N = 10000$)

Parameter	True Value	Euler		LE	
		Mean Bias	Root MSE	Mean Bias	Root MSE
κ_v^P	3.0000	0.0250	0.2537	0.0412	0.2582
θ_v^P	0.1000	−0.0001	0.0043	−0.0001	0.0044
σ_v	0.2500	−0.0013	0.0019	0.0000	0.0014
ρ	−0.8000	0.0002	0.0030	−0.0000	0.0030
γ_1	−7.0000	−0.0773	0.5432	−0.0912	0.5445
γ_2	−6.0000	−0.0516	0.4532	−0.0526	0.4640

Table 2. Monte Carlo Simulation Heston Model: estimation results using stock prices and variance swaps ($N = 10000$)

Parameter	True Value	Euler		LE	
		Mean Bias	Root MSE	Mean Bias	Root MSE
κ_v^P	3.0000	−0.0067	0.1712	0.0203	0.1726
θ_v^P	0.1000	−0.0002	0.0047	−0.0000	0.0046
σ_v	0.2500	−0.0022	0.0075	0.0007	0.0073
ρ	−0.8000	0.0003	0.0037	0.0000	0.0037
γ_1	−7.0000	−0.1838	1.1314	−0.0251	1.1193
γ_2	−6.0000	−0.1104	0.7027	−0.0283	0.6804

Table 3. Monte Carlo Simulation Two-Factor Model: estimation results using stock prices and latent state variables ($N = 10000$)

Parameter	True Value	Euler		LE	
		Mean Bias	Root MSE	Mean Bias	Root MSE
κ_v^P	3.0000	0.0236	0.2772	0.0213	0.2810
σ_v	0.2500	−0.0014	0.0019	0.0000	0.0013
ρ	−0.8000	0.0004	0.0032	−0.0000	0.0031
κ_m^P	0.3000	0.1269	0.2048	0.1273	0.2053
θ_m^P	0.1000	0.0013	0.0159	0.0013	0.0159
σ_m	0.1000	−0.0001	0.0007	−0.0000	0.0007
γ_1	−7.0000	−0.1538	1.4726	−0.0889	1.4937
γ_2	−6.0000	−0.1047	0.8632	−0.0651	0.8550
γ_3	−1.0000	−0.0293	0.3376	−0.0173	0.3191

Table 4. Monte Carlo Simulation Two-Factor Model: estimation results using stock prices and variance swaps ($N = 10000$)

Parameter	True Value	Euler		LE	
		Mean Bias	Root MSE	Mean Bias	Root MSE
κ_v^P	3.0000	0.0021	0.1940	0.0008	0.1927
σ_v	0.2500	−0.0015	0.0021	−0.0002	0.0015
ρ	−0.8000	0.0001	0.0033	−0.0004	0.0033
κ_m^P	0.3000	0.0158	0.0602	0.0153	0.0608
θ_m^P	0.1000	0.0013	0.0161	0.0014	0.0159
σ_m	0.1000	0.0005	0.0021	0.0005	0.0020
γ_1	−7.0000	−0.0439	1.2425	0.0362	1.2468
γ_2	−6.0000	−0.0465	0.7768	0.0015	0.7630
γ_3	−1.0000	−0.0726	0.5494	−0.0846	0.5563

Table 5. Monte Carlo Simulation Constant Jump-Intensity Two-Factor Model: estimation results using stock prices and variance swaps, and LE estimation method ($N = 10000$)

Parameter	True Value	LE	
		Mean Bias	Root MSE
κ_v^P	3.0000	0.1850	0.4051
σ_v	0.2500	-0.0060	0.0062
ρ	-0.8000	0.0260	0.0265
κ_m^P	0.3000	0.0420	0.1487
θ_m^P	0.1000	-0.0059	0.0884
σ_m	0.1000	0.0011	0.0049
γ_1	-7.0000	5.5661	7.2405
γ_2	-6.0000	-0.9059	1.7150
γ_3	-1.0000	0.5954	1.6650
λ	6.0000	-1.3604	1.6328
μ_j^P	-0.0100	-0.0041	0.0123
μ_j^Q	-0.2000	-0.0272	0.0374
σ_j	0.0400	0.0198	0.0209

Table 6. Monte Carlo Simulation Stochastic Jump-Intensity Two-Factor Model: estimation results using stock prices and variance swaps, and LE estimation method ($N = 10000$)

Parameter	True Value	LE	
		Mean Bias	Root MSE
κ_v^P	3.0000	0.3372	0.6688
σ_v	0.2500	0.0169	0.0202
ρ	-0.8000	0.0111	0.0146
κ_m^P	0.3000	0.1111	0.2460
θ_m^P	0.1000	0.0312	0.8054
σ_m	0.1000	0.0107	0.0144
γ_1	-7.0000	6.1806	8.4395
γ_2	-6.0000	-0.8454	2.2510
γ_3	-1.0000	0.9085	2.1713
λ_1	10.0000	-4.9889	5.3182
μ_j^P	-0.0100	-0.0067	0.0178
μ_j^Q	-0.2000	-0.0438	0.0693
σ_j	0.0400	0.0272	0.0419
λ_0	4.0000	-1.1651	1.4749

Variance swap rates							
Maturity	Mean	Std	Skew	Kurt	AC1	Q_{22}	ADF
2	22.14	8.18	1.53	7.08	0.982	62,908.97	−3.79
3	22.32	7.81	1.32	6.05	0.988	66,449.22	−3.52
6	22.87	7.40	1.10	4.97	0.992	69,499.72	−3.30
12	23.44	6.88	0.80	3.77	0.994	71,644.69	−2.82
24	23.93	6.48	0.57	2.92	0.995	72,878.68	−2.47

Table 7. Summary statistics of the variance swap rates on the S&P 500 index at different maturities (in months) from January 4, 1996 to September 2, 2010 for a total of 3,624 observations for each maturity. The table reports mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), first order autocorrelation (AC1) the Ljung–Box portmanteau test for up to 22nd order autocorrelation, Q_{22}, 10% critical value is 30.81; the augmented Dickey–Fuller test for unit root involving 22 augmentation lags, a constant term and time trend, ADF, 10% critical value is -3.16.

Table 8. Parameter Estimates for Heston Model using Variance Swap Rates and S&P 500 Index

	VS without errors				VS with uncorrelated errors				VS with correlated errors			
Parameter	Euler		LE		Euler		LE		Euler		LE	
	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.
κ_v^P	1.9654	0.4551	1.5916	0.2698	0.9567	0.3716	0.8751	0.1733	0.9176	0.3622	0.7973	0.1601
θ_v^P	0.0387	0.0091	0.0403	0.0064	0.0463	0.0180	0.0465	0.0092	0.0465	0.0183	0.0462	0.0092
σ_v	0.3459	0.0085	0.3490	0.0091	0.2978	0.0022	0.2852	0.0022	0.2922	0.0021	0.2715	0.0021
ρ	-0.7439	0.0069	-0.7655	0.0080	-0.7127	0.0071	-0.6964	0.0079	-0.7060	0.0073	-0.6740	0.0084
γ_1	-5.4600	2.7883	-4.2377	3.4162	1.3759	2.5407	1.3005	3.1503	1.3973	2.5450	1.3025	3.0838
γ_2	-5.6777	1.5538	-4.5603	0.8031	-1.2740	1.2446	-1.3036	0.6044	-1.2345	1.2355	-1.3223	0.5856
σ_{e_1}					0.0048	0.0000	0.0048	0.0000	0.0062	0.0001	0.0062	0.0001
σ_{e_2}					0.0064	0.0001	0.0064	0.0001	0.0061	0.0001	0.0060	0.0001
σ_{e_3}					0.0114	0.0004	0.0114	0.0004	0.0107	0.0003	0.0106	0.0003
σ_{e_4}					0.0152	0.0006	0.0148	0.0006	0.0140	0.0005	0.0136	0.0005
ρ_e									0.2951	0.0205	0.2875	0.0203
Log-lik at MLE	22067.42		22050.59		59617.44		59662.30		60018.57		60008.39	
Avg. Run Time	2.6		13.7		4.8		10.3		5.6		8.8	

Table 9. Parameter Estimates for Two-Factor Volatility Model using Variance Swap Rates and S&P 500 Index

	VS without errors				VS with uncorrelated errors				VS with correlated errors			
Parameter	Euler		LE		Euler		LE		Euler		LE	
	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.
κ_v^P	4.6504	0.7391	4.3756	0.3932	5.2638	0.7497	5.0239	0.3883	5.3879	0.7649	5.0604	0.4011
σ_v	0.5255	0.0044	0.5210	0.0048	0.5278	0.0042	0.5237	0.0045	0.5321	0.0042	0.5246	0.0045
ρ	-0.7519	0.0074	-0.7560	0.0073	-0.7388	0.0072	-0.7419	0.0074	-0.7372	0.0073	-0.7431	0.0074
κ_m^P	0.4003	0.2331	0.4008	0.1149	0.2497	0.1544	0.2340	0.0697	0.2319	0.1449	0.2206	0.0671
θ_m^P	0.0498	0.0289	0.0489	0.0138	0.0532	0.0329	0.0531	0.0158	0.0528	0.0330	0.0540	0.0165
σ_m	0.1996	0.0027	0.1979	0.0027	0.1629	0.0007	0.1576	0.0006	0.1564	0.0007	0.1544	0.0006
γ_1	2.2737	2.6469	2.7688	2.4382	0.5161	2.6091	0.6668	2.2992	0.5428	2.6191	0.7417	2.3145
γ_2	-0.7928	1.4125	-0.5120	0.7544	-1.9048	1.4210	-1.7915	0.7392	-1.8919	1.4382	-1.8371	0.7638
γ_3	-1.2759	1.1533	-1.5199	0.5579	-0.5255	0.9462	-0.5735	0.4404	-0.5971	0.9261	-0.5483	0.4340
σ_{e_1}					0.0040	0.0000	0.0039	0.0000	0.0040	0.0000	0.0040	0.0000
σ_{e_2}					0.0025	0.0000	0.0025	0.0000	0.0025	0.0000	0.0025	0.0000
σ_{e_3}					0.0029	0.0000	0.0028	0.0000	0.0028	0.0000	0.0028	0.0000
ρ_e									-0.0964	0.0061	-0.0925	0.0061
Log-lik at MLE	36244.29		36264.59		73248.75		73246.82		73281.82		73274.19	
Avg. Run Time	3.1		8.6		4.3		26.9		3.5		42.6	

Table 10. Parameter Estimates for Constant Intensity Model using Variance Swap Rates and S&P 500 Index

	VS without errors				VS with uncorrelated errors				VS with correlated errors			
Parameter	Euler		LE		Euler		LE		Euler		LE	
	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.
κ_v^P	2.8367	0.5738	2.8108	0.3543	3.8478	0.6236	3.8408	0.3520	4.8575	0.6387	4.7581	0.3629
σ_v	0.4092	0.0022	0.4091	0.0039	0.4323	0.0022	0.4426	0.0041	0.4369	0.0022	0.4451	0.0041
ρ	-0.7317	0.0078	-0.7303	0.0087	-0.7180	0.0079	-0.7226	0.0086	-0.7188	0.0077	-0.7259	0.0084
κ_m^P	0.6939	0.2308	0.5934	0.1579	0.4076	0.1254	0.3901	0.0863	0.2325	0.1264	0.2215	0.0853
θ_m^P	0.0368	0.0121	0.0426	0.0112	0.0267	0.0083	0.0278	0.0062	0.0468	0.0255	0.0481	0.0186
σ_m	0.2059	0.0027	0.2035	0.0038	0.1475	0.0006	0.1473	0.0013	0.1475	0.0006	0.1459	0.0013
γ_1	0.0123	3.6460	0.0112	3.5947	0.0111	3.5868	0.0143	3.4859	-2.4679	3.6288	-2.6391	3.4878
γ_2	-0.0587	1.4030	-0.0546	0.8727	-0.0704	1.4411	-0.0577	0.8024	-2.1926	1.4596	-1.9960	0.8179
γ_3	-2.8009	1.0979	-2.2821	0.7617	-1.7792	0.8488	-1.6666	0.5853	-0.6082	0.8573	-0.5623	0.5853
λ_0	4.1234	1.1019	5.9694	0.7758	5.8623	1.1838	4.2593	0.5945	4.4889	0.9881	5.2727	0.7178
μ_j^P	0.0030	0.0108	0.0082	0.0070	0.0104	0.0065	0.0104	0.0068	0.0108	0.0082	0.0106	0.0055
μ_j^Q	-0.0069	0.0078	-0.0053	0.0052	-0.0047	0.0053	-0.0060	0.0079	-0.0033	0.0071	-0.0035	0.0063
σ_j	0.0454	0.0059	0.0379	0.0025	0.0316	0.0030	0.0360	0.0026	0.0353	0.0038	0.0326	0.0023
σ_{e_1}					0.0039	0.0000	0.0039	0.0000	0.0040	0.0000	0.0040	0.0000
σ_{e_2}					0.0025	0.0000	0.0025	0.0000	0.0025	0.0000	0.0025	0.0000
σ_{e_3}					0.0028	0.0000	0.0029	0.0000	0.0028	0.0000	0.0028	0.0000
ρ_e									-0.0858	0.0060	-0.0891	0.0058
Log-lik at MLE	37397.80		37427.41		74310.04		74328.13		74347.29		74362.36	
Avg. Run Time	7.4		21.2		7.9		10.9		7.3		17.6	

Table 11. Parameter Estimates for Stochastic Intensity Model using Variance Swap Rates and S&P 500 Index

	VS without errors				VS with uncorrelated errors				VS with correlated errors			
Parameter	Euler		LE		Euler		LE		Euler		LE	
	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.	Estimate	Std.Er.
κ_v^P	2.9481	0.5213	2.8012	0.3374	4.4448	0.5971	4.8549	0.3440	5.7587	0.5803	4.8031	0.3528
σ_v	0.3704	0.0082	0.3741	0.0099	0.4028	0.0068	0.4243	0.0089	0.4029	0.0067	0.4185	0.0093
ρ	-0.7219	0.0084	-0.7187	0.0097	-0.7139	0.0081	-0.7232	0.0092	-0.7106	0.0082	-0.7134	0.0097
κ_m^P	0.0884	0.2290	0.1459	0.1587	0.2124	0.1244	0.2295	0.0841	0.1987	0.1251	0.2335	0.0862
θ_m^P	0.2554	0.6611	0.1558	0.1687	0.0452	0.0266	0.0441	0.0162	0.0485	0.0307	0.0428	0.0159
σ_m	0.1894	0.0043	0.1906	0.0049	0.1385	0.0019	0.1422	0.0022	0.1388	0.0018	0.1414	0.0023
γ_1	-0.7217	4.3172	0.3954	3.2922	-1.2622	4.9264	-2.1988	4.0228	-4.6224	4.3951	-2.5453	4.2060
γ_2	-0.3478	1.4074	-0.0051	0.9085	-1.4280	1.4798	-2.2990	0.8179	-4.5720	1.4325	-2.2442	0.8512
γ_3	-0.2206	1.1844	-0.2737	0.8194	-0.5285	0.8974	-0.6658	0.5910	-0.4579	0.9017	-0.6731	0.6100
λ_1	49.6958	20.1884	40.0291	11.3265	62.9733	21.0223	15.2104	7.2403	57.0516	21.7805	44.7703	17.2272
μ_j^P	0.0033	0.0116	0.0094	0.0083	0.0128	0.0107	0.0105	0.0156	-0.0050	0.0091	0.0096	0.0075
μ_j^Q	-0.0224	0.0123	-0.0422	0.0106	-0.0068	0.0122	-0.0113	0.0215	-0.0159	0.0125	-0.0008	0.0092
σ_j	0.0469	0.0062	0.0370	0.0060	0.0424	0.0051	0.0545	0.0055	0.0408	0.0055	0.0384	0.0030
λ_0	3.0766	0.9287	2.6935	0.5493	2.6971	0.6711	1.6636	0.3304	2.6364	0.8048	3.6685	0.6208
σ_{e_1}					0.0039	0.0000	0.0039	0.0000	0.0040	0.0000	0.0040	0.0000
σ_{e_2}					0.0025	0.0000	0.0025	0.0000	0.0025	0.0000	0.0025	0.0000
σ_{e_3}					0.0028	0.0000	0.0028	0.0000	0.0028	0.0000	0.0028	0.0000
ρ_e									-0.0909	0.0060	-0.0876	0.0057
Log-lik at MLE	37426.66		37454.37		74334.82		74330.47		74369.37		74381.83	
Avg. Run Time	12.5		47.1		7.8		15.3		9.5		18.7	

Table 12. Instantaneous Recovered quantities (in %'s) from Heston Model via Euler

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	20.1722	7.2315	0.6653	0.6141	8.5512	51.8260
DRP	7.2203	5.2753	1.8662	5.9201	1.1498	42.2332
VRP	-1.6566	1.2104	-1.8662	5.9201	-9.6899	-0.2638
Out of Sample						
$\sqrt{\widehat{v}_t}$	27.2188	10.3313	1.2800	1.5473	11.0325	66.8261
DRP	13.3253	11.2317	2.1180	4.4362	1.9139	70.2184
VRP	-3.0573	2.5770	-2.1180	4.4362	-16.1107	-0.4391

Table 13. Instantaneous Recovered quantities (in %'s) from Heston Model via LE

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	20.2218	7.1119	0.6816	0.6492	8.8870	51.5181
DRP	7.3609	5.2972	1.8662	5.9201	1.2653	42.5191
VRP	-1.6497	1.1872	-1.8662	5.9201	-9.5291	-0.2836
Out of Sample						
$\sqrt{\widehat{v}_t}$	27.1662	10.2119	1.2891	1.5640	11.2638	66.3947
DRP	13.4913	11.2784	2.1180	4.4362	2.0325	70.6205
VRP	-3.0236	2.5276	-2.1180	4.4362	-15.8271	-0.4555

Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$.

Table 14. Instantaneous Recovered quantities (in %'s) from 2-Factor Volatility Model via Euler

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	19.5696	7.4057	1.1796	1.8590	4.8686	60.1068
$\sqrt{\widehat{m}_t}$	22.2696	6.6548	0.6652	0.0512	12.2137	50.5925
DRP	7.1906	5.9925	2.4526	8.6494	0.3893	59.3391
VRP	-4.4069	3.6726	-2.4526	8.6494	-36.3671	-0.2386
LRMRP	-0.5046	0.3088	-1.3892	2.6226	-2.3907	-0.1393
Out of Sample						
$\sqrt{\widehat{v}_t}$	26.8830	11.6522	1.7012	2.6264	12.0338	75.3682
$\sqrt{\widehat{m}_t}$	27.5180	7.6822	0.0705	-0.5851	12.4737	45.1535
DRP	14.0972	14.3834	2.5130	6.3082	2.3785	93.2975
VRP	-8.6397	8.8152	-2.5130	6.3082	-57.1791	-1.4577
LRMRP	-0.7623	0.4041	-0.6280	-0.3261	-1.9043	-0.1453

Table 15. Instantaneous Recovered quantities (in %'s) from 2-Factor Volatility Model via LE

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	19.6288	7.3507	1.1615	1.8207	6.8673	59.6100
$\sqrt{\widehat{m}_t}$	22.3377	6.6411	0.6772	0.0621	12.3428	50.5445
DRP	7.4564	6.1347	2.4300	8.5308	0.8005	60.3115
VRP	-4.2339	3.4834	-2.4300	8.5308	-34.2461	-0.4545
LRMRP	-0.4598	0.2805	-1.3924	2.5905	-2.1629	-0.1290
Out of Sample						
$\sqrt{\widehat{v}_t}$	26.9068	11.5457	1.6904	2.5924	12.4979	74.8550
$\sqrt{\widehat{m}_t}$	27.5343	7.6288	0.0487	-0.6135	12.5738	44.7227
DRP	14.5479	14.6735	2.4982	6.2295	2.6512	95.1051
VRP	-8.2606	8.3319	-2.4982	6.2295	-54.0027	-1.5054
LRMRP	-0.6911	0.3627	-0.5931	-0.4002	-1.6933	-0.1339

Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

Table 16. Instantaneous Recovered quantities (in %'s) from Constant Intensity Model via Euler

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	17.8758	7.9541	0.9606	1.3288	2.0873	58.7799
$\sqrt{\widehat{m}_t}$	21.0105	7.0636	0.6129	-0.0667	9.9583	50.0551
DRP	1.4678	1.3758	2.4158	8.4578	0.0167	13.2481
JRP	6.3849				6.3849	6.3849
VRP	-3.6670	3.4373	-2.4158	8.4578	-33.0986	-0.0417
LRMRP	-0.4407	0.2989	-1.3951	2.5658	-2.2474	-0.0890
Out of Sample						
$\sqrt{\widehat{v}_t}$	25.6478	11.8580	1.6179	2.3916	10.0679	74.1004
$\sqrt{\widehat{m}_t}$	26.4528	7.9882	-0.0369	-0.5909	10.2188	43.8352
DRP	3.0608	3.2835	2.4892	6.1816	0.3887	21.0541
JRP	6.3849				6.3849	6.3849
VRP	-7.6470	8.2034	-2.4892	6.1816	-52.6009	-0.9710
LRMRP	-0.6848	0.3830	-0.5659	-0.4562	-1.7236	-0.0937

Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

Table 17. Instantaneous Recovered quantities (in %'s) from Constant Intensity Model via LE

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	17.8645	7.9599	0.9548	1.3168	2.5859	58.7255
$\sqrt{\widehat{m}_t}$	21.0174	7.0523	0.6153	-0.0638	9.9947	50.0178
DRP	0.7662	0.7184	2.4126	8.4414	0.0134	6.9090
JRP	7.4618				7.4618	7.4618
VRP	-3.3982	3.1861	-2.4126	8.4414	-30.6407	-0.0594
LRMRP	-0.4032	0.2731	-1.3955	2.5618	-2.0527	-0.0820
Out of Sample						
$\sqrt{\widehat{v}_t}$	25.6421	11.8533	1.6153	2.3843	10.0267	74.0503
$\sqrt{\widehat{m}_t}$	26.4460	7.9720	-0.0387	-0.5946	10.2498	43.7523
DRP	1.5984	1.7137	2.4871	6.1708	0.2014	10.9854
JRP	7.4618				7.4618	7.4618
VRP	-7.0886	7.6000	-2.4871	6.1708	-48.7190	-0.8932
LRMRP	-0.6259	0.3494	-0.5616	-0.4650	-1.5706	-0.0862

Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

Table 18. Instantaneous Recovered quantities (in %'s) from Stochastic Intensity Model via Euler

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	17.1606	7.4845	0.9843	1.3793	3.0256	55.8867
$\sqrt{\widehat{m}_t}$	20.0851	6.6387	0.6226	-0.0489	9.7615	47.5054
DRP	3.3684	3.1129	2.4154	8.4557	0.0880	30.0173
JRP	4.9781	1.9843	2.4154	8.4557	2.8870	21.9653
VRP	-6.4566	5.9668	-2.4154	8.4557	-57.5372	-0.1686
LRMRP	-0.2844	0.1902	-1.3948	2.5680	-1.4345	-0.0606
Out of Sample						
$\sqrt{\widehat{v}_t}$	24.4874	11.2276	1.6248	2.4097	9.8166	70.4346
$\sqrt{\widehat{m}_t}$	25.2100	7.5225	-0.0253	-0.5965	10.0025	41.6478
DRP	6.9729	7.4286	2.4889	6.1802	0.9261	47.6790
JRP	7.2758	4.7353	2.4889	6.1802	3.4213	33.2236
VRP	-13.3656	14.2392	-2.4889	6.1802	-91.3911	-1.7752
LRMRP	-0.4399	0.2439	-0.5684	-0.4512	-1.1026	-0.0636

Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\mathrm{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0 + \lambda_1 v_t)$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\mathrm{LRMRP}_t = \gamma_3\sigma_m m_t$.

Table 19. Instantaneous Recovered quantities (in %'s) from Stochastic Intensity Model via LE

	Mean	Std	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\sqrt{\widehat{v}_t}$	17.4006	7.6698	0.9672	1.3427	3.0934	56.8958
$\sqrt{\widehat{m}_t}$	20.4281	6.8054	0.6200	-0.0560	9.8209	48.4616
DRP	1.2691	1.1808	2.4118	8.4376	0.0336	11.3618
JRP	5.5314	1.5757	2.4118	8.4376	3.8827	18.9996
VRP	-3.3964	3.1599	-2.4118	8.4376	-30.4062	-0.0899
LRMRP	-0.4414	0.2972	-1.3956	2.5611	-2.2358	-0.0918
Out of Sample						
$\sqrt{\widehat{v}_t}$	24.9025	11.4589	1.6187	2.3930	9.8461	71.7357
$\sqrt{\widehat{m}_t}$	25.6686	7.6972	-0.0351	-0.5983	10.0637	42.3886
DRP	2.6369	2.8162	2.4867	6.1684	0.3403	18.0617
JRP	7.3566	3.7581	2.4867	6.1684	4.2919	27.9402
VRP	-7.0567	7.5367	-2.4867	6.1684	-48.3361	-0.9106
LRMRP	-0.6836	0.3801	-0.5607	-0.4668	-1.7106	-0.0964

Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0 + \lambda_1 v_t)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$.

Table 20. Pricing Errors (in volatility %'s units) from Heston Model via Euler

	Mean	RMSE	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\widehat{VS}_{2m} - VS_{2m}$	-0.0934	0.8530	-0.4111	2.2968	-4.7736	3.0354
$\widehat{VS}_{6m} - VS_{6m}$	0.0235	1.1375	-1.2178	3.2298	-4.6151	4.9912
$\widehat{VS}_{24m} - VS_{24m}$	0.9350	3.0767	-0.9065	0.0665	-8.0448	7.1408
Out of Sample						
$\widehat{VS}_{2m} - VS_{2m}$	0.2774	1.4121	-3.5967	19.6498	-10.4229	3.2534
$\widehat{VS}_{6m} - VS_{6m}$	-0.3134	1.3909	1.5269	7.0353	-2.9798	10.3361
$\widehat{VS}_{24m} - VS_{24m}$	0.1022	3.0224	0.3366	-0.2379	-6.8087	11.8545

Table 21. Pricing Errors (in volatility %'s units) from Heston Model via LE

	Mean	RMSE	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\widehat{VS}_{2m} - VS_{2m}$	-0.0808	0.8508	-0.5148	2.4388	-4.8256	2.9651
$\widehat{VS}_{6m} - VS_{6m}$	0.0018	1.1190	-1.0708	3.3361	-4.5599	5.0923
$\widehat{VS}_{24m} - VS_{24m}$	1.0014	2.9503	-0.9457	0.2279	-7.8017	8.5185
Out of Sample						
$\widehat{VS}_{2m} - VS_{2m}$	0.2594	1.4204	-3.6314	19.7978	-10.5395	3.2285
$\widehat{VS}_{6m} - VS_{6m}$	-0.2581	1.4034	1.6553	7.5622	-2.9267	10.6361
$\widehat{VS}_{24m} - VS_{24m}$	0.4690	3.0741	0.4664	0.2638	-6.4846	13.3749

Table 22. Pricing Errors (in volatility %'s units) from Stochastic Intensity Model via Euler

	Mean	RMSE	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\widehat{VS}_{2m} - VS_{2m}$	-0.1831	0.7434	-0.5063	2.1194	-4.1869	2.9676
$\widehat{VS}_{6m} - VS_{6m}$	0.0633	0.3947	-1.7668	14.2941	-3.6369	2.4515
$\widehat{VS}_{24m} - VS_{24m}$	0.1519	0.5601	-0.5073	1.8494	-3.0744	3.3492
Out of Sample						
$\widehat{VS}_{2m} - VS_{2m}$	0.2195	1.0026	-4.7349	30.8543	-8.5755	3.0098
$\widehat{VS}_{6m} - VS_{6m}$	-0.2517	0.4759	3.1368	34.3960	-1.5872	4.5505
$\widehat{VS}_{24m} - VS_{24m}$	0.1427	0.5563	-0.4892	1.6882	-2.2120	1.9287

Table 23. Pricing Errors (in volatility %'s units) from Stochastic Intensity Model via LE

	Mean	RMSE	Skew.	Exc. Kurt.	Min	Max
In Sample						
$\widehat{VS}_{2m} - VS_{2m}$	-0.1828	0.7429	-0.5108	2.1308	-4.1932	2.9636
$\widehat{VS}_{6m} - VS_{6m}$	0.0639	0.3944	-1.7645	14.2417	-3.6271	2.4630
$\widehat{VS}_{24m} - VS_{24m}$	0.1428	0.5603	-0.5231	1.8533	-3.1328	3.3393
Out of Sample						
$\widehat{VS}_{2m} - VS_{2m}$	0.2181	1.0046	-4.7387	30.8483	-8.5946	2.9979
$\widehat{VS}_{6m} - VS_{6m}$	-0.2494	0.4750	3.1959	34.9625	-1.5709	4.5776
$\widehat{VS}_{24m} - VS_{24m}$	0.1287	0.5511	-0.5258	1.7237	-2.2251	1.8900

References

Aït-Sahalia, Y., 1999, "Transition Densities for Interest Rate and Other Nonlinear Diffusions," *Journal of Finance*, 54, 1361–1395.

——— , 2002a, "Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach," *Econometrica*, 70, 223–262.

——— , 2002b, "Telling from Discrete Data Whether the Underlying Continuous-Time Model is a Diffusion," *Journal of Finance*, 57, 2075–2112.

——— , 2008, "Closed-Form Likelihood Expansions for Multivariate Diffusions," *Annals of Statistics*, 36, 906–937.

Aït-Sahalia, Y., M. Karaman, and L. Mancini, 2012, "Variance Swaps, Risk Premiums, and Expectation Hypothesis," .

Aït-Sahalia, Y., and R. Kimmel, 2007, "Maximum Likelihood Estimation of Stochastic Volatility Models," *Journal of Financial Economics*, 83, 413–452.

——— , 2010, "Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions," *Journal of Financial Economics*, 98, 113144.

Alizadeh, S., M. Brandt, and F. Diebold, 2002, "Range-Based Estimation of Stochastic Volatility Models," *Journal of Finance*, 57, 1047–1091.

Andersen, T., L. Benzoni, and J. Lund, 2002, "An Empirical Investigation of Continuous-Time Equity Return Models," *Journal of Finance*, 57, 1239–1284.

Bates, D. S., 2000, "Post-'87 Crash Fears in the S&P 500 Futures Option Market," *Journal of Econometrics*, 94, 181–238.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–654.

Chernov, M., R. Gallant, E. Ghysels, and G. Tauchen, 2003, "Alternative Models of Stock Price Dynamics," *Journal of Econometrics*, 116, 225 – 257.

Cox, J. C., 1975, "The Constant Elasticity of Variance Option Pricing Model," *The Journal of Portfolio Management, Special Issue published in 1996*, pp. 15–17.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, “A Theory of the Term Structure of Interest Rates,” *Econometrica*, 53, 385–408.

Delbaen, F., and W. Schachermayer, 1994, “A General Version of the Fundamental Theorem of Asset Pricing,” *Mathematische Annalen*, 300, 463–520.

Egloff, D., M. Leippold, and L. Wu, 2010, “The Term Structure of Variance Swap Rates and Optimal Variance Swap Investments,” *Journal of Financial and Quantitative Analysis*, 45, 1279–1310.

Jensen, B., and R. Poulsen, 2002, “Transition Densities of Diffusion Processes: Numerical Comparison of Approximation Techniques,” *Journal of Derivatives*, 9, 18–32.

Pan, J., 2002, “The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study,” *Journal of Financial Economics*, 63, 3–50.

Protter, P., 2005, *Stochastic Integration and Differential Equations*. Springer-Verlag, New York, second edn.

Todorov, V., 2010, “Variance Risk Premium Dynamics: The Role of Jumps,” *Review of Financial Studies*, 23, 345–383.

Vasicek, O., 1977, “An Equilibrium Characterization of the Term Structure,” *Journal of Financial Economics*, 5, 177–188.

Part V

Curriculum Vitae

MUSTAFA ARIF KARAMAN

Email : mustafa.karaman@bf.uzh.ch
Phone: + 41 44 634 39 54
Date / Place of Birth : 02.10.1983 / Turkey

University of Zurich
Institute of Banking and Finance
Plattenstr. 32,CH-8032
Zurich, Switzerland

EDUCATION

- **September 2007 – April 2012**
  Ph.D. Candidate in Finance, University of Zürich, Switzerland.
  Advisor: Prof. Loriano Mancini.
- **October 2005 - July 2007**
  M.Sc. in Finance, HEC - University of Lausanne, Switzerland.
- **September 2001- June 2005**
  B.Sc. in Statistics, Middle East Technical University, Ankara, Turkey.

PUBLICATION (REFEREED JOURNAL)

- "The Impact of Terrorism on Financial Markets: An Empirical Study" (with Marc Chesney and Ganna Reshetar), *Journal of Banking and Finance,* volume 35, pages 253-267, 2011.

WORKING PAPERS

- "Variance Swaps, Risk Premiums, and Expectation Hypothesis" (with Yacine Aït-Sahalia and Loriano Mancini), 2011.
- "Euler Approximation and Likelihood Expansion for Continuous-Time Derivative Pricing Models: A comparative analysis", 2011-2012.

TEACHING EXPERIENCE

- *Sept-Dec 2009 & Sept-Dec 2010*: Teaching Assistant; Advanced Topics in Financial Econometrics Course; Financial Engineering Master Program at Swiss Federal Institute of Technology – Lausanne (EPFL).
- *Sept-Dec 2009 & Sept-Dec 2010*: Teaching Assistant; Econometrics Course; Financial Engineering Master Program at Swiss Federal Institute of Technology – Lausanne (EPFL).
- *Jan-Apr2007*: Teaching Assistant; Finance Fundamentals Course; Executive MBA Program at HEC – University of Lausanne.

PROFESSIONAL EXPERIENCE

- *July 2004, Ankara – Turkey:* Internship at the Banking and Financial Institutions at the Central Bank of the Republic of Turkey

CONFERENCES ATTENDED

- *October 2011, Lausanne:* Swissquote Conference on Asset Management.
- *May 2011, Lausanne*: Princeton-Lausanne Workshop on Quantitative Finance.
- *October 2010, Lausanne*: Swissquote Conference on Interest Rate and Credit Risk.
- *October 2010, Paris*: HEC-Paris Finance and Statistics Conference.
- *June 2009, Geneva*: SoFiE European Conference in Geneva.
- *June 2009, Bern*: Gerzensee Doctoral Workshop.
- *March 2009, Berlin*: Recent Developments in Measuring and Modeling Financial Market Volatility.
- *2007-2011, Zurich*: University of Zurich Finance Seminars & Brown Bag Lunch Seminars.
- *October 2006, Lucerne*: Structured Products Switzerland Conference.